Elementary Theory and Methods for Elliptic Partial Differential Equations

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Contents

CHAPTER 1

Introduction and Basic Theory

In this introductory chapter, we provide some preliminary background which we will use later in establishing various results for general elliptic partial differential equations (PDEs). The material found within these notes aims to compile the fundamental theory for second-order elliptic PDEs and serves as complementary notes to many well-known references on the subject, c.f., [6, 7, 9, 13, 16]. Several recommended resources on basic background that supplement these notes and the aforementioned references are the textbooks [3, 10, 28].

We will mainly focus on the Dirichlet problem,

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{1.1}$$

where U is a bounded open subset of $\mathbb{R}^n$ with boundary ∂U, and $u : \mathbb{R}^n \mapsto \mathbb{R}$ is the unknown quantity. For this problem, $f : U \mapsto \mathbb{R}$ is given, and L is a second-order differential operator having either the form

$$Lu = -\sum_{i,j=1}^{n} D_j \left(a^{ij}(x) D_i u \right) + \sum_{i=1}^{n} b^i(x) D_i u + c(x)u, \tag{1.2}$$

or else

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u + \sum_{i=1}^{n} b^i(x) D_i u + c(x)u, \tag{1.3}$$

for given coefficient functions a^{ij}, b^i, and c $(i, j = 1, 2, \ldots, n)$ which are assumed to be measurable in $\bar{U}$, the closure of the set U. However, in this chapter, we take these coefficients to be continuous in $\bar{U}$. If L takes the form (1.2), then it is said to be in **divergence form**, and if it takes the form (1.3), then it is said to be in **non-divergence form**.

Remark 1.1. *Here, $D_{ij} = D_i D_j$. In practice,* (1.2) *is natural for energy methods while (1.3) is more appropriate for the maximum principles. In addition, the Dirichlet problem* (1.1) *can be extended to systems, i.e., $Lu_i = f_i$ in U, and $u_i = 0$ on ∂U, for $i = 1, 2, \ldots, L \in \mathbb{Z}^+$. A simple example of a second-order differential operator is the Laplacian, $L := -\Delta$, where $a^{ij} = \delta_{ij}$, $b^i = c = 0$ $(i, j = 1, 2 \ldots, n)$ in either* (1.2) *or* (1.3).

Remark 1.2. *The elliptic theory for equations in divergence form was developed first as we can easily exploit the distributional framework and energy methods for weak solutions in Sobolev spaces, for example. Much of our focus in these notes will be on establishing the basic elliptic PDE theory for equations in divergence form.*

Remark 1.3. *Extending this theory to elliptic equations in non-divergence form has certain obstacles, and its treatment requires a somewhat different approach. We shall study one way of examining such equations using another concept of a weak solution called a viscosity solution, which are defined with the help of maximum and comparison principles. We shall give a brief introduction to fully nonlinear elliptic equations in non-divergence form and their viscosity solutions in Chapter 4.*

Unless stated otherwise, we shall always assume that L is **uniformly elliptic**, i.e., there exist $\lambda, \Lambda > 0$ such that

$$\lambda|\xi|^2 \;\le\; \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \;\le\; \Lambda|\xi|^2 \text{ a.e } x \in U, \text{ for all } \xi \in \mathbb{R}^n.$$

Moreover, $u \in H_0^1(U)$ is said to be a *weak solution* of (1.1) in divergence form if

$$B[u,\, v] = (f,\, v), \text{ for all}\, v \in H_0^1(U),$$

where $B[\cdot,\, \cdot]$ is the associated bilinear form,

$$B[u,\, v] := \int_U \sum_{i,j=1}^{n} a^{ij} D_i u D_j v + \sum_{i=1}^{n} b^i(x) D_i u v + c(x) u v \, dx.$$

1.1 Harmonic Functions

First we shall introduce the mean-value property, which provides the key ingredient in establishing many important properties for harmonic functions.

1.1.1 Mean Value Properties

Definition 1.1. *For $u \in C(U)$ we define* *(i) u satisfies the first mean value property (in U) if*

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, d\sigma_y \quad \textit{for any } B_r(x) \subset U;$$

(ii) u satisfies the second mean value property if

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy \quad \textit{for any } B_r(x) \subset U.$$

Remark 1.4. *These two definitions are equivalent. To see this, observe that if we rewrite (i) as*

$$u(x) r^{n-1} = \frac{1}{\omega_n} \int_{\partial B_r(x)} u(y)\, d\sigma_y,$$

where ω_n denotes the surface area of the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$, i.e., $\omega_n = n\alpha(n) = n|B_1(0)|$ where $B_1(0) \subset \mathbb{R}^n$ is the n-dimensional unit ball centered at the origin, then integrate with respect to r, we get

$$u(x)\frac{r^n}{n} = \frac{1}{\omega_n} \int_0^r \int_{\partial B_s(x)} u(y)\, d\sigma_y\, ds = \frac{1}{\omega_n} \int_{B_r(x)} u(y)\, dy.$$

If we rewrite (ii) as

$$u(x) r^n = \frac{n}{\omega_n} \int_{B_r(x)} u(y)\, dy = \frac{n}{\omega_n} \int_0^r \int_{\partial B_s(x)} u(y)\, d\sigma_y ds$$

then differentiate with respect to r, we obtain (i).

Remark 1.5. *The mean value properties can easily be expressed in the following ways.*

(i) $u \in C(U)$ satisfies the first mean value property if

$$u(x) = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x + r\omega)\, d\sigma_\omega \quad \textit{for any } B_r(x) \subset U;$$

(ii) $u \in C(U)$ satisfies the second mean value property if

$$u(x) = \frac{n}{\omega_n} \int_{B_1(0)} u(x + ry)\, dy \quad \textit{for any } B_r(x) \subset U;$$

Theorem 1.1. *If $u \in C^2(U)$ is harmonic, then u satisfies the mean value property.*

Proof. Set

$$\phi(r) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\, d\sigma_y = \frac{1}{\omega_n}\int_{\partial B_1(0)} u(x + r\omega)\, d\sigma_\omega.$$

Then

$$\begin{aligned}
\phi'(r) &= \frac{1}{\omega_n}\int_{\partial B_1(0)} Du(x + r\omega)\cdot\omega\, d\sigma_\omega = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} Du(y)\cdot\frac{y-x}{r}\, d\sigma_y \\
&= \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}(y)\, d\sigma_y = \frac{r}{n}\frac{n}{\omega_n r^n}\int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}(y)\, d\sigma_y \\
&= \frac{r}{n}\frac{1}{|B_r(x)|}\int_{B_r(x)} \Delta u(y)\, dy = 0.
\end{aligned}$$

Hence, ϕ is constant. Therefore, by the Lebesgue differentiation theorem (see Theorem 3.4),

$$\phi(r) = \lim_{t\longrightarrow 0}\phi(t) = \lim_{t\longrightarrow 0}\frac{1}{|\partial B_t(x)|}\int_{\partial B_t(x)} u(y)\, d\sigma_y = u(x).$$

□

The next theorem is the converse of the previous result. Namely, functions satisfying the mean value property are harmonic.

Theorem 1.2. *If $u \in C^2(U)$ satisfies the mean value property, then u is harmonic.*

Proof. If $\Delta u \not\equiv 0$, we may assume without loss of generality that there exists a ball $B_r(x) \subset U$ for which $\Delta u > 0$ within $B_r(x)$ However, as in the previous computation,

$$0 = \phi'(r) = \frac{r}{n}\frac{1}{|B_r(x)|}\int_{B_r(x)} \Delta u(y)\, dy > 0,$$

which is a contradiction. □

The next theorem is the maximum principle for harmonic functions.

Theorem 1.3 (Strong maximum principle for harmonic functions)**.** *Suppose $u \in C^2(U) \cap C(\bar{U})$ is harmonic within U.*

(i) Then

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

(ii) In addition, if U is connected and there exists a point $x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u(x),$$

then u is constant in U.

Proof. Suppose that there is such a point $x_0 \in U$ with $u(x_0) = M := \max_{\bar{U}} u$. Then for $0 < r < dist(x_0, \partial U)$, the mean value property asserts

$$M = u(x_0) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(y)\, dy \leq M.$$

Hence, equality holds only if $u \equiv M$ in $B_r(x_0)$. That is, the set $\{x \in U \,|\, u(x) = M\}$ is both open and relatively closed in U. Therefore, this set must equal U since U is connected. This proves assertion (ii), from which (i) follows. □

1.1.2 Sub-harmonic and Super-harmonic Functions

Interestingly, mean-value properties and maximum principles hold for sub-harmonic and super-harmonic functions. Let us state such results including some important applications. We say a function $u \in C^2(U)$ is sub-harmonic in U if $-\Delta u \leq 0$ in U and super-harmonic if $-\Delta u \geq 0$ in U.

Lemma 1.1 (Mean Value Inequality). *Let $x \in B_{r_0}(x) \subset U$ for some $r_0 > 0$.*

(i) If $-\Delta u > 0$ within $B_{r_0}(x)$, then for any $r \in (0, r_0)$,

$$u(x) > \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, d\sigma_y.$$

It follows that if x_0 is a minimum point of u in U, then

$$-\Delta u(x_0) \leq 0.$$

(ii) If $-\Delta u < 0$ within $B_{r_0}(x)$, then for any $r \in (0, r_0)$,

$$u(x) < \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, d\sigma_y.$$

It follows that if x_0 is a maximum point of u in U, then

$$-\Delta u(x_0) \geq 0.$$

Proof. As in the proof of Theorem 1.1, we see

$$\int_{B_r(x)} \Delta u(x)\, dx = r^{n-1} \int_{\partial B_1(0)} \frac{\partial u}{\partial r}(x + r\omega)\, d\sigma_\omega. \tag{1.4}$$

We only prove (i) since the proof of (ii) follows from similar arguments. From (1.4), we see that if $-\Delta u > 0$, then

$$\frac{\partial}{\partial r} \int_{\partial B_1(0)} u(x + r\omega)\, d\sigma_\omega < 0.$$

Integrating this from 0 to r yields

$$\int_{\partial B_1(0)} u(x+r\omega)\, d\sigma_\omega - u(x)|\partial B_1(0)| < 0,$$

in which the desired inequality follows immediately. To prove the second statement in (i), we proceed by contradiction. On the contrary, suppose that x_0 is a minimum point of u in U and assume that $-\Delta u(x_0) > 0$. By the continuity of u, we can find a $\delta > 0$ for which $-\Delta u > 0$ within $B_\delta(x_0)$. But the mean value inequality implies that

$$u(x_0) > \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} u(y)\, d\sigma_y \quad \text{for any} \quad r \in (0, \delta).$$

This contradicts with the assumption that x_0 is a minimum of u. $\square$

A nice application of the mean value inequalities is the weak maximum principle for the Laplacian. Analogous results for more general uniformly elliptic equations are provided below. In addition, unlike the strong maximum principles for harmonic functions provided earlier, we do not make any connectedness assumption on the domain U.

Theorem 1.4 (Weak Maximum Principle for the Laplacian)**.** *Suppose that $u \in C^2(U) \cap C(\bar{U})$.*

(i) If

$$-\Delta u \geq 0 \quad \textit{within} \quad U,$$

then

$$\min_{\bar{U}} u \geq \min_{\partial U} u.$$

(ii) If

$$-\Delta u \leq 0 \quad \textit{within} \quad U,$$

then

$$\max_{\bar{U}} u \leq \max_{\partial U} u.$$

Proof. We only prove (i) since (ii) follows from similar arguments. First, we assume u is strictly super-harmonic: $-\Delta u > 0$ within U. Let x_0 be a minimum of u in U, but the mean value inequality implies $-\Delta u(x_0) \leq 0$, which is a contradiction. Thus, $\min_{\bar{U}} u \geq \min_{\partial U} u$. Now, suppose u is super-harmonic: $-\Delta u \geq 0$ within U and set $u_\epsilon = u - \epsilon|x|^2$. Obviously, u_ϵ is strictly super-harmonic, i.e.,

$$-\Delta u_\epsilon = -\Delta u + 2\epsilon n > 0.$$

It follows that $\min_{\bar{U}} u_\epsilon \geq \min_{\partial U} u_\epsilon$ and the desired result follows after sending $\epsilon \longrightarrow 0$. $\square$

An application of the weak maximum principle is the following interior gradient estimate for harmonic functions.

Corollary 1.1 (Bernstein)**.** *Suppose u is harmonic in U and let $V \subset\subset U$. Then there holds*

$$\sup_V |Du| \le C \sup_{\partial U} |u|,$$

where $C = C(n, V)$ is a positive constant. In particular, for any $\alpha \in (0, 1)$ there holds

$$|u(x) - u(y)| \le C|x - y|^\alpha \sup_{\partial U} |u| \quad \textit{for any} \quad x, y \in V.$$

Proof. A direct calculation shows

$$\Delta(|Du|^2) = 2\sum_{i,j=1}^n (D_{ij}u)^2 + 2\sum_{i=1}^n D_i u D_i(\Delta u) = 2\sum_{i,j=1}^n (D_{ij}u)^2 \ge 0. \tag{1.5}$$

That is, $|Du|^2$ is a sub-harmonic function in U. Then, for any test function $\varphi \in C_0^1(U)$, a basic identity yields

$$\Delta(\varphi|Du|^2) = (\Delta\varphi)|Du|^2 + 2D\varphi \cdot D(|Du|^2) + \varphi\Delta(|Du|^2).$$

Hence, combining this with (1.5) gives us

$$\Delta(\varphi|Du|^2) = (\Delta\varphi)|Du|^2 + 4\sum_{i,j=1}^n D_i\varphi D_j u D_{ij}u + 2\varphi\sum_{i,j=1}^n (D_{ij}u)^2.$$

We establish the gradient estimates using a cutoff function. By taking $\varphi = \eta^2$ for some $\eta \in C_0^1(U)$ with $\eta \equiv 1$ within V, we obtain by Hölder's inequality,

$$\begin{aligned}\Delta(\eta^2|Du|^2) &= 2\eta\Delta\eta|Du|^2 + 2|D\eta|^2|Du|^2 + 8\eta\sum_{i,j=1}^n D_i\eta D_j u D_{ij}u + 2\eta^2\sum_{ij=1}^n (D_{ij}u)^2 \\ &\ge (2\eta\Delta\eta - 6|D\eta|^2)|Du|^2 \ge -C|Du|^2 = -\frac{C}{2}\Delta(u^2),\end{aligned}$$

where C is a positive constant depending only on η. In the last line, we used the fact that $\Delta(u^2) = 2|Du|^2 + 2u\Delta u = 2|Du|^2$ since u is harmonic. By choosing $a \ge C/2$ large enough, we obtain

$$\Delta(\eta^2|Du|^2 + au^2) \ge 0.$$

By part (ii) of the weak maximum principle, we obtain

$$\sup_V |Du|^2 \le \sup_V \left\{\eta^2|Du|^2 + a|u|^2\right\} \le \sup_{\bar{U}} \left\{\eta^2|Du|^2 + a|u|^2\right\} = a\sup_{\partial U} |u|^2.$$

□

Theorem 1.5 (Removable Discontinuity)**.** *Let u be a harmonic function in $B_R(0)\backslash\{0\}$ that satisfies $u(x) = o(|x|^{2-n})$ as $|x| \longrightarrow 0$ if $n \geq 3$ or $u(x) = o(\log|x|)$ as $|x| \longrightarrow 0$ if $n = 2$. Then u can be defined at 0 so that it is smooth and harmonic in $B_R(0)$.*

Proof. For simplicity, let us only consider the case $n \geq 3$, since the case when $n = 2$ is treated exactly the same except that the fundamental solution is of the logarithmic type.

Assume u is continuous in the punctured disk $B_R(0)\backslash\{0\}$ and let v solve

$$\begin{cases} \Delta v = 0 & \text{in } B_R(0), \\ v = u & \text{on } \partial B_R(0). \end{cases}$$

Moreover, assume that $\lim_{|x|\to 0} u(x)|x|^{n-2} = 0$, i.e., any possible singularity of u at the origin grows no faster than the fundamental solution $|x|^{2-n}$ (of course, this property is trivial whenever u is bounded).

It suffices to prove that $u \equiv v$ in $B_R(0)\backslash\{0\}$. Set $w = v - u$ in $B_R(0)\backslash\{0\}$, $0 < r < R$, and $M_r := \max_{\partial B_r(0)} |w|$. Clearly,

$$|w(x)| \leq M_r \frac{r^{n-2}}{|x|^{n-2}} \quad \text{on} \quad \partial B_r(0).$$

Note that both w and $\frac{1}{|x|^{n-2}}$ are harmonic in $B_R(0)\backslash B_r(0)$. Hence, the weak maximum principle implies

$$|w(x)| \leq M_r \frac{r^{n-2}}{|x|^{n-2}} \quad \text{for any} \quad x \in B_R(0)\backslash B_r(0).$$

Then for each fixed $x \neq 0$,

$$|w(x)| \leq \max_{\partial B_R(0)} |u| \cdot \frac{r^{n-2}}{|x|^{n-2}} + \underbrace{\frac{\max_{\partial B_r(0)} |u|}{|x|^{n-2}} \cdot r^{n-2}}_{|x|^{2-n} o(1) \text{ as } r\to 0} \longrightarrow 0 \quad \text{as} \quad r \longrightarrow 0,$$

where we used the estimate

$$M_r = \max_{\partial B_r(0)} |v - u| \leq \max_{\partial B_r(0)} |v| + \max_{\partial B_r(0)} |u| \leq \max_{\partial B_R(0)} |v| + \max_{\partial B_r(0)} |u| \leq \max_{\partial B_R(0)} |u| + \max_{\partial B_r(0)} |u|.$$

Hence, $w \equiv 0$ in $B_R(0)\backslash\{0\}$. □

1.1.3 Further Properties of Harmonic Functions

Theorem 1.6 (Regularity)**.** *If $u \in C(U)$ satisfies the mean value property in U, then $u \in C^\infty(U)$.*

Proof. Define $\eta \in C_c^\infty(\mathbb{R}^n)$ to be the standard mollifier

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases}$$

where $C > 0$ is chosen so that $\|\eta\|_{L^1(\mathbb{R}^n)} = 1$, and set $u_\epsilon := \eta_\epsilon * u$ in $U_\epsilon = \{x \in U \,|\, dist(x, \partial U) > \epsilon\}$. Then $u_\epsilon \in C^\infty(U_\epsilon)$. Now, the mean-value property and simple calculations imply

$$\begin{aligned}
u_\epsilon(x) &= \int_U \eta_\epsilon(x-y)u(y)\,dy = \frac{1}{\epsilon^n}\int_{B_\epsilon(x)} \eta\left(\frac{|x-y|}{\epsilon}\right) u(y)\,dy \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right)\left(\int_{\partial B_r(x)} u(y)\,d\sigma_y\right) dr = \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right)\frac{\omega_n}{n} r^{n-1}u(x)\,dr \\
&= u(x)\int_{B_\epsilon(0)} \eta_\epsilon(y)\,dy = u(x).
\end{aligned}$$

Thus, $u \equiv u_\epsilon$ in U_ϵ and so $u \in C^\infty(U_\epsilon)$ for each $\epsilon > 0$. □

Remark 1.6. *We mention some other regularizing properties of the mollifier introduced above. If $u \in C(U)$, then $u_\epsilon \longrightarrow u$ uniformly on compact subsets of U as $\epsilon \longrightarrow 0$. Moreover, if $1 \le p < \infty$ and the function $u \in L^p_{loc}(U)$, then $u_\epsilon \longrightarrow u$ in $L^p_{loc}(U)$.*

Theorem 1.7 (Pointwise Estimates for Derivatives)**.** *Suppose u is harmonic in U. Then*

$$|D^\alpha u(x)| \le \frac{C_k}{r^{n+k}}\|u\|_{L^1(B_r(x))}, \tag{1.6}$$

for each ball $B_r(x) \subset U$ and each multi-index α of order $|\alpha| = k$. Particularly,

$$C_0 = \frac{n}{\omega_n},\ C_k = \frac{(2^{n+1}k)^k n^{k+1}}{\omega_n}\ (k = 1, 2, \ldots). \tag{1.7}$$

Proof. We proceed by induction in which the case when $k = 0$ is clear. For $k = 1$, we note that derivatives of harmonic functions are also harmonic. Consequently,

$$|u_{x_i}(x)| = \left|\frac{1}{|B_{r/2}(x)|}\int_{B_{r/2}(x)} u_{x_i}(y)\,dy\right| = \left|\frac{n2^n}{\omega_n r^n}\int_{\partial B_{r/2}(x)} u(y)\nu_i\,d\sigma_y\right| \le \frac{2n}{r}\|u\|_{L^\infty(\partial B_{r/2}(x))} \tag{1.8}$$

If $y \in \partial B_{r/2}(x)$, then $B_{r/2}(y) \subset B_r(x) \subset U$, and so

$$|u(y)| \le \frac{n}{\omega_n}\left(\frac{2}{r}\right)^n \|u\|_{L^1(B_r(x))},$$

where we used the estimate for the previous case $k = 0$. Inserting this into estimate (1.8) completes the verification for the case $k = 1$. Now assume that $k \ge 2$ and the estimates (1.6)–(1.7) hold for all balls in U and for each multi-index of order less than or equal to $k - 1$. Fix $B_r(x) \subset U$ and let α be a multi-index with $|\alpha| = k$. Then $D^\alpha u = (D^\beta u)_{x_i}$ for some $i \in \{1, 2, \ldots, n\}$, $|\beta| = k - 1$. Using similar calculations as before, we obtain

$$|D^\alpha u(x)| \le \frac{nk}{r}\|D^\beta u\|_{L^\infty(\partial B_{r/k}(x))}.$$

If $y \in B_{r/k}(x)$, then $B_{\frac{k-1}{k}r}(y) \subset B_r(x) \subset U$. Thus, estimates (1.6)–(1.7) imply

$$|D^\beta u(y)| \leq \frac{n(2^{n+1}n(k-1))^{k-1}}{\omega_n(\frac{k-1}{k}r)^{n+k-1}}\|u\|_{L^1(B_r(x))}.$$

Combining the last two estimates imply the desired estimate

$$|D^\alpha u(x)| \leq \frac{n(2^{n+1}nk)^k}{\omega_n r^{n+k}}\|u\|_{L^1(B_r(x))} = \frac{C_k}{r^{n+k}}\|u\|_{L^1(B_r(x))}.$$

□

Theorem 1.8 (Liouville). *Suppose $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is harmonic and bounded. Then u is constant.*

Proof. Fix $x \in \mathbb{R}^n$, $r > 0$, and apply Theorem 1.7 on $B_r(x)$ to get

$$|Du(x)| \leq \frac{C_1}{r^{n+1}}\|u\|_{L^1(B_r(x))} \leq \frac{C_1}{r^{n+1}}\frac{\omega_n}{n}r^n\|u\|_{L^\infty(B_r(x))} \leq \frac{C}{r} \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty.$$

Hence, $Du \equiv 0$, and so u is constant. □

Theorem 1.9 (Harnack's Inequality). *For each connected open set $V \subset\subset U$, there exists a positive constant $C = C(V)$, depending only on V, such that*

$$\sup_V u \leq C \inf_V u$$

for all non-negative harmonic functions u in U. In particular,

$$C^{-1}u(y) \leq u(x) \leq Cu(y)$$

for all $x, y \in V$.

Remark 1.7. *Harnack's inequality asserts that non-negative harmonic functions within V are in a sense all comparable and shows that the oscillation of such functions can be controlled. Basically, a harmonic function cannot be small (large, respectively) at some point in V unless it is small (large,respectively) on all other points in V.*

Proof. Let $r := \frac{1}{4}dist(V, \partial U)$ and choose $x, y \in V$ with $|x - y| \leq r$. Then

$$\begin{aligned} u(x) &= \frac{1}{|B_{2r}(x)|}\int_{B_{2r}(x)} u(z)\,dz \geq \frac{n}{\omega_n 2^n r^n}\int_{B_r(y)} u(z)\,dz \\ &= \frac{1}{2^n}\frac{1}{|B_r(y)|}\int_{B_r(y)} u(z)\,dz = \frac{1}{2^n}u(y). \end{aligned}$$

Hence, $\frac{1}{2^n}u(y) \leq u(x) \leq 2^n u(y)$ if $x, y \in V$ with $|x - y| \leq r$. Since V is connected and its closure is compact, we can cover $\bar{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$, each of which has radius $r/2$ and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 2, 3, \ldots N$. Then

$$u(x) \geq \frac{1}{2^{n(N+1)}}u(y)$$

for all $x, y \in V$. □

The following provides an another equivalent characterization of harmonic functions, and it gives a proper motivation for the notion of viscosity solutions to fully nonlinear elliptic equations (see Chapter 4).

Theorem 1.10. *Let U be a open bounded domain in $\mathbb{R}^n$. Then, u is a harmonic function in U if and only if u is continuous and satisfies the following two conditions.*

(i) If $u-\varphi$ has a local maximum at $x_0 \in U$ and $\varphi \in C^2(U)$, then $-\Delta\varphi(x_0) \leq 0$.

(ii) If $u-\varphi$ has a local minimum at $x_0 \in U$ and $\varphi \in C^2(U)$, then $-\Delta\varphi(x_0) \geq 0$.

Proof. If u is harmonic in U, then u is clearly continuous and showing it satisfies the two conditions is obvious. For instance, if $u-\varphi$ has a local maximum at $x_0 \in U$, then

$$-\Delta\varphi(x_0) = \Delta(u(x_0) - \varphi(x_0)) \leq 0.$$

The second condition is verified in a similar manner. Now suppose that the two conditions are satisfied. By regularity properties of harmonic functions as indicated earlier, we may assume that u is C^2. Then, it is clear that if $u \in C^2(U)$, then we can set $\varphi = u$ in the two conditions and conclude that $u = \varphi$ is harmonic in U. □

1.1.4 Energy and Comparison Methods for Harmonic Functions

The following are simple approaches for harmonic functions that we will make use of in the later chapters. We begin with Cacciopolli's inequality, which is sometimes called the reversed Poincaré inequality.

Lemma 1.2 (Cacciopolli's Inequality)**.** *Suppose $u \in C^1(B_1)$ satisfies*

$$\int_{B_1} a^{ij}(x) D_i u D_j \varphi \, dx = 0 \quad \textit{and} \quad \varphi \in C_0^1(B_1).$$

Then for any function $\eta \in C_0^1(B_1)$, we have

$$\int_{B_1} \eta^2 |Du|^2 \, dx \leq C \int_{B_1} |D\eta|^2 u^2 \, dx,$$

where $C = C(\lambda, \Lambda)$ is a positive constant.

Proof. For any $\eta \in C_0^1(B_1)$ set $\varphi = \eta^2 u$. From the definition of a weak solution, we have

$$\lambda \int_{B_1} \eta^2 |Du|^2 \, dx \leq \Lambda \int_{B_1} \eta |u| |D\eta| |Du| \, dx.$$

Then by Hölder's inequality,

$$\begin{aligned}\lambda \int_{B_1} \eta^2 |Du|^2 \, dx &\leq \Lambda \int_{B_1} \eta |u| |D\eta| |Du| \, dx \\ &\leq \Lambda \left(\int_{B_1} \eta^2 |Du|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B_1} |D\eta|^2 u^2 \, dx \right)^{\frac{1}{2}}\end{aligned}$$

and the result follows immediately. □

Corollary 1.2. *Let u be as in Lemma 1.2. Then for any $0 \leq r < R \leq 1$, there holds*

$$\int_{B_r} |Du|^2 \, dx \leq \frac{C}{(R-r)^2} \int_{B_R} |u|^2 \, dx,$$

where $C = C(\lambda, \Lambda)$.

Proof. Choose η such that $\eta \equiv 1$ on B_r, $\eta \equiv 0$ outside B_R and $|D\eta| \leq 2(R-r)^{-1}$ then apply Lemma 1.2. □

Corollary 1.3. *Let u be as in Lemma 1.2. Then for any $0 < R \leq 1$, there hold*

$$\int_{B_{R/2}} u^2 \, dx \leq \theta \int_{B_R} u^2 \, dx, \quad \textit{and} \quad \int_{B_{R/2}} |Du|^2 \, dx \leq \theta \int_{B_R} |Du|^2 \, dx,$$

where $\theta = \theta(n, \lambda, \Lambda) \in (0, 1)$.

Proof. Take $\eta \in C_0^1(B_R)$ with $\eta \equiv 1$ on $B_{R/2}$ and $|D\eta| \leq 2R^{-1}$. Then by Lemma 1.2 and since $D\eta \equiv 0$ in $B_{R/2}$, we have

$$\int_{B_R} |D(\eta u)|^2 \, dx \leq \int_{B_R} |D\eta|^2 u^2 + \eta^2 |Du|^2 \, dx \leq C \int_{B_R} |D\eta|^2 u^2 \, dx \leq \frac{C}{R^2} \int_{B_R \setminus B_{R/2}} u^2 \, dx.$$

From this estimate and Poincaré's inequality, we obtain

$$\int_{B_{R/2}} u^2 \, dx \leq \int_{B_R} (\eta u)^2 \, dx \leq C_n R^2 \int_{B_R} |D(\eta u)|^2 \, dx \leq C \int_{B_R \setminus B_{R/2}} u^2 \, dx.$$

This further implies

$$(C+1) \int_{B_{R/2}} u^2 \, dx \leq C \int_{B_R} u^2 \, dx,$$

which completes the proof of the first estimate. The proof of the second estimate follows similar arguments. □

Remark 1.8. *Interestingly, Corollary 1.3 implies that every harmonic function in $\mathbb{R}^n$ with finite L^2-norm are identically zero and every harmonic function in $\mathbb{R}^n$ with finite Dirichlet integral is constant. Moreover, iterating the estimates in Corollary 1.3 leads to the following estimates. Let u be as in Lemma 1.2, then for any $0 < \rho < r \le 1$ there hold*

$$\int_{B_\rho} u^2\,dx \le C\left(\frac{\rho}{r}\right)^\mu \int_{B_r} u^2\,dx, \quad \textit{and} \quad \int_{B_\rho} |Du|^2\,dx \le C\left(\frac{\rho}{r}\right)^\mu \int_{B_r} |Du|^2\,dx,$$

for some positive constant $\mu = \mu(n, \lambda, \Lambda)$. Later on we prove that we can take $\mu \in (n-2, n)$.

Lemma 1.3 (Basic Estimates for Harmonic Functions)**.** *Suppose $\{a^{ij}\}$ is a constant positive definite matrix satisfying the uniformly elliptic condition,*

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \textit{for any} \quad \xi \in \mathbb{R}^n \tag{1.9}$$

for some $0 < \lambda \le \Lambda$. Suppose $u \in C^1(B_1)$ satisfies

$$\int_{B_1} a^{ij}(x) D_i u D_j \varphi = 0 \quad \textit{for any} \quad \varphi \in C_0^1(B_1).$$

Then for any $0 < \rho \le r$, there hold

$$\int_{B_\rho} |u|^2\,dx \le C\left(\frac{\rho}{r}\right)^n \int_{B_r} |u|^2\,dx,$$

$$\int_{B_\rho} |u - (u)_{0,\rho}|^2\,dx \le C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |u - (u)_{0,r}|^2\,dx,$$

where $C = C(\lambda, \Lambda)$.

Proof. By dilation we may assume that $r = 1$. We restrict our attention to the range $\rho \in (0, 1/2]$, since the estimates are trivial by a change of variables when $\rho \in (1/2, 1]$.

Claim: There holds

$$\|u\|^2_{L^\infty(B_{1/2})} + \|Du\|^2_{L^\infty(B_{1/2})} \le C(\lambda, \Lambda) \int_{B_1} |u|^2\,dx.$$

From this we get the first result,

$$\int_{B_\rho} |u|^2\,dx \le \rho^n \|u\|^2_{L^\infty(B_{1/2})} \le \rho^n(\|u\|^2_{L^\infty(B_{1/2})} + \|Du\|^2_{L^\infty(B_{1/2})}) \le c\rho^n \int_{B_1} |u|^2\,dx.$$

The claim also implies, using Poincaré's inequality on balls (see Theorem A.24), the inequality

$$\int_{B_\rho} |u - (u)_{0,\rho}|^2\,dx \le \rho^{n+2} \|Du\|^2_{L^\infty(B_{1/2})} \le c\rho^{n+2} \int_{B_1} |u|^2\,dx.$$

If u is a solution of (1.9) then so is $u - (u)_{0,1}$. With u replaced by $u - (u)_{0,1}$ in the above inequality, we readily obtain the second result,

$$\int_{B_\rho} |u - (u)_{0,\rho}|^2 \, dx \leq c\rho^{n+2} \int_{B_1} |u - (u)_{0,1}|^2 \, dx.$$

It only remains to prove the claim. If u is a solution of (1.9), then so are any derivatives of u. By applying Corollary 1.2 to the derivatives of u, we conclude that for any positive integer k

$$\|u\|_{H^k(B_{1/2})} \leq c(k, \lambda, \Lambda)\|u\|_{L^2(B_1)}.$$

By fixing k sufficiently large, the Sobolev embedding theorem implies that $H^k(B_{1/2}) \hookrightarrow C^1(\bar{B}_{1/2})$. Thus,

$$\|u\|_{C^1(\bar{B}_{1/2})} = \sup_{\bar{B}_{1/2}} |u(x)| + \sup_{\bar{B}_{1/2}} |Du(x)| \leq c(n)\|u\|_{H^k(B_{1/2})} \leq c(n, k, \lambda, \Lambda)\|u\|_{L^2(B_1)}.$$

This completes the proof of the lemma. □

1.2 The Classical Maximum Principles

In this section, we consider an elliptic operator L in non-divergence form:

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u,$$

where the coefficients a^{ij}, b^i, c are continuous in some bounded open subset $U \subset \mathbb{R}^n$ and the uniform ellipticity condition holds. We now introduce the important maximum principles for second-order uniformly elliptic equations. In the next chapter, we will instead focus on uniformly elliptic operators in divergence form, which are more appropriate for the energy and variational methods introduced in that chapter. In the later chapters, we will also look at maximum principles for weak solutions when we study the weak Harnack inequality and its connection with regularity properties of solutions to elliptic equations (see Theorem 3.36 for example).

1.2.1 The Weak Maximum Principle

Theorem 1.11 (Weak Maximum Principle)**.** *Assume $u \in C^2(U) \cap C(\bar{U})$ and $c \equiv 0$ in U.*

(a) If $Lu \leq 0$ in U, then $\max_{\bar{U}} u = \max_{\partial U} u$.

(b) If $Lu \geq 0$ in U, then $\min_{\bar{U}} u = \min_{\partial U} u$.

Proof. We prove assertion (a).

Step 1: First we assume $Lu < 0$ in U but there exists $x_0 \in U$ such that $u(x_0) = \max_{\bar{U}} u$. Of course, at this maximum point there hold

$$(i)\ Du(x_0) = 0 \ \text{ and } \ (ii)\ D^2u(x_0) \leq 0. \tag{1.10}$$

Since $A = (a^{ij}(x_0))$ is symmetric and positive definite, there is an orthogonal matrix $O = (o_{ij})$ such that

$$OAO^T = diag(d_1, d_2, \ldots, d_n), \tag{1.11}$$

where $OO^T = I$ and $d_k > 0$ for $k = 1, 2, \ldots, n$. Write $y = x_0 + O(x - x_0)$ so that $x - x_0 = O^T(y - x_0)$,

$$u_{x_i} = \sum_{k=1}^{n} u_{y_k} o_{ki} \ \text{ and } \ u_{x_i x_j} = \sum_{k,\ell=1}^{n} u_{y_k y_\ell} o_{ki} o_{\ell j} \ (i, j = 1, 2, \ldots, n).$$

Hence, at the point x_0,

$$\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{n} a^{ij} u_{y_k y_\ell} o_{ki} o_{\ell j} = \sum_{k=1}^{n} d_k u_{y_k y_k} \leq 0, \tag{1.12}$$

where in the last line the inequality is due to (1.10)(ii) and the fact that $d_k > 0$ for $k = 1, 2, \ldots, n$, and the equality is due to (1.11). From (1.10)(i) and (1.12), at the point x_0 we have

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} \geq 0,$$

and we arrive at a contradiction.

Step 2: Now we complete the proof for the case when $Lu \geq 0$ in U. Set

$$u^\epsilon(x) := u(x) + \epsilon e^{\lambda x_1},\ x \in U,$$

where $\lambda > 0$ will be specified below and $\epsilon > 0$. From the uniform ellipticity condition, there holds $a^{ii}(x) \geq \theta$ for $i = 1, 2, \ldots, n$, $x \in U$. Hence,

$$Lu^\epsilon = Lu + \epsilon L(e^{\lambda x_1}) \leq \epsilon e^{\lambda x_1}(-\lambda^2 a^{11} + \lambda b^1) \leq \epsilon e^{\lambda x_1}(-\lambda^2\theta + \|b\|_{L^\infty}\lambda) < 0 \ \text{ in } U,$$

provided that $\lambda > 0$ is chosen to be sufficiently large. Namely, we have $Lu^\epsilon > 0$ in U and we conclude $\max_{\bar{U}} u^\epsilon = \max_{\partial U} u^\epsilon$ from step 1. Let $\epsilon \longrightarrow 0$ to find $\max_{\bar{U}} u = \max_{\partial U} u$.

Assertion (b) follows easily from (a) once we make the simple observation that $-u$ is a subsolution, i.e., $L(-u) \leq 0$ in U whenever u is a supersolution. □

Remark 1.9. *Maximum principles for elliptic equations such as the previous weak maximum principle typically come in two parts, e.g., parts (a) and (b). Although the term "maximum" principle should technically only refer to a statement like part (a), we shall adopt the standard convention that something like part (b) will also be referred to as a "maximum" principle. Indeed, the direct relationship between (a) and (b) can be seen in the proof above. In view of this, we sometimes only state and prove one part, e.g., part (a), of a maximum principle but the reader should be aware a corresponding second part, e.g., part (b), will hold as well.*

A simple extension of the weak maximum principle is the following.

Theorem 1.12 (Weak Maximum Principle for $c \geq 0$)**.** *Assume $u \in C^2(U) \cap C(\bar{U})$ and $c \geq 0$ in U, and $f \in C(\bar{U})$. If $Lu \leq f$ in U, then*

$$\max_U u \leq C \max_{\partial U} u^+ + \max_{\bar{U}} |f|,$$

where $C > 0$ depends on n, λ, $\max_{\bar{U}} |b^i|$ and $diam(\Omega)$.

Consequently, if $u \in C^2(U) \cap C(\bar{U})$ is a solution of $Lu = f$ in Ω, then

$$\|u\|_{L^\infty(U)} \leq C' \|u\|_{L^\infty(\partial U)} + \|f\|_{L^\infty(\bar{U})},$$

where $C' > 0$ depends on the same quantities as the previous constant C.

Remark 1.10. *The term "weak" from the above weak maximum principles come from their obvious implication that "the non-negative maximum can always be attained on the boundary," which is weaker than the statement "if u is non-constant, then it cannot attain its non-negative maximum in the interior of the domain." The latter statement is called the strong maximum principle, which is the next result we look at.*

1.2.2 The Strong Maximum Principle

Just as we have for harmonic functions, the weak maximum principles may be strengthened after some added conditions on U. In order to do this, we make use of Hopf's Lemma.

Lemma 1.4 (Hopf's Lemma)**.** *Assume $u \in C^2(U) \cap C(\bar{U})$ and $c \equiv 0$ in U. Suppose further that $Lu \leq 0$ in U and there is a ball B contained in U with a point $x^0 \in \partial U \cap \partial B$ such that*

$$u(x) < u(x^0) \text{ for all } x \in B. \tag{1.13}$$

(a) Then

$$\frac{\partial u}{\partial \nu}(x^0) > 0, \tag{1.14}$$

where ν is the outer unit normal to B at x_0.

(b) If $c \geq 0$ in U, the same conclusion holds provided $u(x^0) \geq 0$.

Remark 1.11. *(a) If $U = B$ and $u \in C^2(B) \cap C(B \cup \{x^0\})$, we can actually prove that for any outward direction ν such that $\nu \cdot n(x^0) > 0$, we have that*

$$\liminf_{t \to 0^+} \frac{u(x^0) - u(x^0 - t\nu)}{t} > 0.$$

So, under the stronger regularity assumption that $u \in C^2(B) \cap C^1(B \cup \{x^0\})$, the directly yields (1.14).

(b) An analogous result holds for when $Lu \leq 0$ in U but with the inequalities in the above "interior ball" condition and the conclusions are switched to be in the opposite direction, i.e.,

$$\frac{\partial u}{\partial \nu}(x^0) < 0.$$

Proof of Hopf's Lemma. Assume $c \geq 0$ and also assume, without loss of generality, that $B = B_r(0)$ for some $r > 0$.
Step 1: Define

$$v(x) := e^{-\lambda |x|^2} - e^{-\lambda r^2} \text{ for } x \in B_r(0)$$

for $\lambda > 0$ to be specified below. Then, from the uniform ellipticity condition,

$$\begin{aligned} Lv = & -\sum_{i,j=1}^n a^{ij} v_{x_i x_j} + \sum_{i=1}^n b^i v_{x_i} + cv \\ &= e^{-\lambda|x|^2} \sum_{i,j=1}^n a^{ij}(-4\lambda^2 x_i x_j + 2\lambda\delta_{ij}) - e^{-\lambda|x|^2} \sum_{i=1}^n b^i 2\lambda x_i + c(e^{-\lambda|x|^2} - e^{-\lambda r^2}) \\ &\leq e^{-\lambda|x|^2}(-4\theta\lambda^2|x|^2 + 2\lambda tr(A) + 2\lambda|b||x| + c), \end{aligned}$$

for $A = (a^{ij})$ and $b = (b^i)$. Next consider the open annulus $R = B_r^0(0) \backslash B_{r/2}(0)$ and so

$$Lv \leq e^{-\lambda|x|^2}(-\theta\lambda^2 r^2 + 2\lambda tr(A) + 2\lambda|b|r + c) \leq 0 \text{ in } R \tag{1.15}$$

provided that $\lambda > 0$ is fixed to be large enough.

Step 2: In view of (1.13), there exists a constant $\epsilon > 0$ small for which

$$u(x^0) \geq u(x) + \epsilon v(x) \text{ for } x \in \partial B_{r/2}(0). \tag{1.16}$$

In addition, notice since $v \equiv 0$ on $\partial B_r(0)$,

$$u(x^0) \geq u(x) + \epsilon v(x) \text{ for } x \in \partial B_r(0). \tag{1.17}$$

Step 3: From (1.15), we see

$$L(u + \epsilon v - u(x^0)) \leq -cu(x^0) \leq 0 \text{ in } R,$$

and from (1.16) and (1.17) we have

$$u + \epsilon v - u(x^0) \le 0 \text{ on } \partial R.$$

The weak maximum principle implies that $u+\epsilon v-u(x^0) \le 0$ in R, but $u(x^0)+\epsilon v(x^0)-u(x^0) = 0$, and so

$$\frac{\partial u}{\partial \nu}(x^0) + \epsilon \frac{\partial v}{\partial \nu}(x^0) \ge 0.$$

Consequently,

$$\frac{\partial u}{\partial \nu}(x^0) \ge -\epsilon \frac{\partial v}{\partial \nu}(x^0) = -\frac{\epsilon}{r} Dv(x^0) \cdot x^0 = 2\lambda \epsilon r e^{-\lambda r^2} > 0.$$

This completes the proof. □

Theorem 1.13 (Strong Maximum Principle)**.** *Assume $u \in C^2(U) \cap C(\bar{U})$, $c \equiv 0$ in $U \subset \mathbb{R}^n$, and U is connected, open and bounded.*

(a) If $Lu \le 0$ in U and u attains its maximum over $\bar{U}$ at an interior point, then u is constant within U.

(b) If $Lu \ge 0$ in U and u attains its minimum over $\bar{U}$ at an interior point, then u is constant within U.

Proof. We prove statement (a) only, since statement (b) follows similarly. Write $M = \max_{\bar{U}} u$ and take $C = \{x \in U \,|\, u(x) = M\}$. If C is empty or if $u \equiv M$ we are done. Otherwise, if $u \not\equiv M$, set

$$V = \{x \in U \,|\, u(x) < M\}.$$

Choose a point $y \in V$ satisfying $dist(y, C) < dist(y, \partial U)$ and let B denote the largest ball with center y whose interior lies in V. Then there exists some point $x^0 \in C$ with $x^0 \in \partial B$. It is easy to check that V satisfies the interior ball condition at x^0. Hence, by part (a) of Hopf's lemma, $\partial u/\partial \nu(x^0) > 0$. But this contradicts with the fact that $Du(x^0) = 0$ since u attains its maximum at $x^0 \in U$. □

If the coefficient $c(x)$ is non-negative, then we have the following version of the strong maximum principle. Its proof is the same as before but invokes statement (b) in Hopf's lemma.

Theorem 1.14 (Strong Maximum Principle for $c \ge 0$)**.** *Assume $u \in C^2(U) \cap C(\bar{U})$, $c \ge 0$ in $U \subset \mathbb{R}^n$, and U is connected, open and bounded.*

(a) If $Lu \le 0$ in U and u attains a non-negative maximum over $\bar{U}$ at an interior point, then u is constant within U.

(b) If $Lu \ge 0$ in U and u attains a non-positive minimum over $\bar{U}$ at an interior point, then u is constant within U.

A simple but useful consequence of the strong maximum principle is the following comparison principle.

Corollary 1.4. *If U is connected, open and bounded, $c \geq 0$ and suppose $u \in C^2(U) \cap C(\bar{U})$ satisfies*

$$\begin{cases} Lu \leq 0 & \text{in } U \\ u \leq 0 & \text{on } \partial U, \end{cases}$$

then either $u \equiv 0$ or $u < 0$ in U.

Finally, we state a quantitative version of the maximum principle for second-order elliptic equations called Harnack's inequality. However, a more general version with proof shall be offered in Chapter 3. There we will see the importance of Harnack's inequality and how it applies to obtaining several results on a weaker notion of solution, called weak or distributional solutions, for elliptic equations. This includes results on their regularity properties, Liouville type theorems, and even a version of the strong maximum principle adapted to weak solutions.

Theorem 1.15. *Assume u is a non-negative C^2 solution of*

$$Lu = 0 \ \text{ in } U,$$

and suppose $V \subset\subset U$ is connected. Then there exists a constant C such that

$$\sup_V u \leq C \inf_V u.$$

The constant C depends only on V and the coefficients of L.

1.2.3 Some Refinements and Extensions

The earlier comparison principle of Corollary 1.4 can be generalized by removing the non-negative assumption $c \geq 0$ in U.

Theorem 1.16. *If U is connected, open and bounded and suppose $u \in C^2(U) \cap C(\bar{U})$ satisfies*

$$\begin{cases} Lu \leq 0 & \text{in } U, \\ u \leq 0 & \text{on } \partial U, \end{cases}$$

then either $u \equiv 0$ or $u < 0$ in U.

Proof. By writing $c = c^+ - c_-$, where $c^+ = \max\{c, 0\}$ and $c^- = \max\{0, -u\}$ are the positive and negative parts of c, we get

$$\tilde{L}u := -\sum_{i,j=1}^{n} a^{ij}(x)u + \sum_{i=1}^{n} b^i(x)D_i u + c^+(x)u \leq c^-(x)u \leq 0.$$

Thus, applying the strong maximum principle for $c \geq 0$ to the subsolution $u \in C^2(U) \cap C(\bar{U})$ of

$$\begin{cases} \tilde{L}u \leq 0 & \text{in } U, \\ u \leq 0 & \text{on } \partial U, \end{cases}$$

we arrive at the desired conclusion. $\square$

The maximum principle and Hopf's lemma can be adapted to handle unbounded domains with non-positive lower-order terms. We provide a version below that will be useful for our purposes in later chapters; specifically when we study the method of moving planes.

Theorem 1.17. *Let U be a domain in $\mathbb{R}^n$ with smooth boundary ∂U, and assume $u \in C^2(U) \cap C(\bar{U})$ satisfies*

$$\begin{cases} -\Delta u + \sum_{i=1}^n b^i(x) D_i u + c(x) u \geq 0 & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{1.18}$$

where $b^i(x)$ and $c(x)$ are bounded functions. Then the following hold.

(a) If u vanishes at some point in U, then $u \equiv 0$ in U;

(b) If u is non-trivial in U, then $\partial u / \partial \nu < 0$ on ∂U.

Proof. Part (a) Proceeding by contradiction, we assume u vanishes at some point in U but $u \not\equiv 0$ in U. Therefore, we can consider the positive part of our domain U,

$$U^+ = \{x \in U \mid u(x) > 0\},$$

which is obviously a non-empty open subset with C^2 boundary ∂U^+, and $\partial U^+ \backslash \partial U \neq \emptyset$. Note that $u = 0$ on ∂U^+, and we can pick $x^0 \in \partial U^+ \backslash \partial U$. We may choose suitably small $R > 0$ and $x^1 \in U^+$ such that $x^0 \in \partial B_{R/4}(x^1) \subset U^+$. Then $\partial B_{R/4}(x^1) \subset B_R(x^0)$.

Let $\lambda_R = \lambda(B_R(x^0))$ denote the first eigenvalue for the Dirichlet Laplacian on $B_R(x^0)$ and suppose φ_R a corresponding positive eigenfunction (see Section 2.3 for more details). By a simple scaling and translation argument, it is easy to see $\lambda_R = \lambda_1 / R^2$, where $\lambda_1 = \lambda_1(B_1(0))$, and if φ_1 is a corresponding positive eigenfunction for the first eigenvalue λ_1, we can just take $\varphi_R(x) = \varphi(x/R)$.

Set $w = u/\varphi_R$. Obviously, x^0 is an interior point in U and a local minimum of w in $B_R(x^0)$ and so

$$Dw(x^0) = 0. \tag{1.19}$$

To finish our proof of this part, we verify that w is indeed a supersolution to some suitable second-order equation, then we apply Hopf's Lemma to deduce a contradiction with the vanishing of the gradient at x^0. A simple calculation reveals

$$-\Delta w + \underbrace{(-2w) \cdot \frac{Dw}{w} + \sum_{i=1}^n b^i(x) D_i w(x)}_{=\sum_{i=1}^n \tilde{b}^i(x) D_i w(x)} + \tilde{c}(x) w \geq 0 \text{ in } U,$$

where, in $B_R(x^0)$, there holds

$$\tilde{c}(x) = \frac{-\Delta\varphi_R}{\varphi_R} + \frac{1}{R}\sum_{i=1}^{n}\frac{b^i(x)D_i\varphi_R(x)}{\varphi_R(x)} + c(x) = \frac{\lambda_1}{R^2} + \frac{1}{R}\sum_{i=1}^{n}\frac{b^i(x)D_i\varphi_R(x)}{\varphi_R(x)} + c(x).$$

By the boundedness of the coefficients b^i and c, we can choose $R > 0$ small enough so that $\tilde{c} > 0$ in $B_R(x^0)$. Furthermore, $w(x) > 0$ in $B_R(x^0)$ and $w(x^0) = u(x^0)/\varphi_R(x^0) = 0$, so the interior ball condition holds at x^0 and we can apply Hopf's Lemma to conclude that

$$\frac{\partial w}{\partial \nu}(x^0) < 0.$$

But this contradicts with (1.19), and this completes the proof of part (a).

Part (b) The proof of this part is essentially the same as the proof of part (a), except we choose $x^0 \in \partial U$ and take an open ball $B_{R/2}(x^1) \subset U$ such that $x^0 \in \partial B_{R/2}(x^1)$. Since φ_R can be taken to be radially symmetric and decreasing about the center x^0, we have

$$\varphi_R(0) = \max_{B_R(x^0)} \varphi_R(x) \quad \text{and} \quad \frac{\partial \varphi_R}{\partial \nu}(x^0) = 0.$$

Hence,

$$\frac{\partial u}{\partial \nu}(x^0) = \varphi_R(x^0)\frac{\partial w}{\partial \nu}(x^0) + w(x^0)\frac{\partial \varphi_R}{\partial \nu}(x^0) < 0.$$

□

We give a useful comparison principle that we will be essential in developing the methods of Chapter 5.

Theorem 1.18 (Maximum principle based on comparisons)**.** *Assume that U is a bounded domain. Let ϕ be a positive function on $\bar{U}$ satisfying*

$$-\Delta\phi + \lambda(x)\phi \geq 0. \tag{1.20}$$

Assume that u is a classical solution of

$$\begin{cases} -\Delta u + c(x)u \geq 0 & \text{in } U, \\ u \geq 0 & \text{on } \partial U. \end{cases} \tag{1.21}$$

If

$$c(x) > \lambda(x) \ \text{ for all } \ x \in U, \tag{1.22}$$

then

$$u \geq 0 \ \text{ in } \ U.$$

If U is unbounded, then the result remains true provided that the following additional condition is assumed:

$$\liminf_{|x|\longrightarrow\infty} \frac{u(x)}{\phi(x)} \geq 0. \tag{1.23}$$

Proof. We proceed by contradiction. Let $v(x) = u(x)/\phi(x)$ and assume that $u < 0$ at some point in U. Thus, $v < 0$ at that same point, since ϕ is positive in U. Let $x^0 \in U$ be the minimum of v and by a simple calculation, we obtain that

$$-\Delta v = 2Dv \cdot \frac{D\phi}{\phi} + \frac{1}{\phi}(-\Delta u + \frac{\Delta\phi}{\phi}u). \tag{1.24}$$

However, since x^0 is a minimum of v, we have that

$$-\Delta v(x^0) \leq 0 \tag{1.25}$$

and

$$Dv(x^0) = 0. \tag{1.26}$$

But from (5.4)–(5.6) and since $u(x^0) < 0$, we have that

$$-\Delta u(x^0) + \frac{\Delta\phi}{\phi}(x^0)u(x^0) \geq -\Delta u(x^0) + \lambda(x^0)u(x^0) > -\Delta u(x^0) + c(x^0)u(x^0) \geq 0.$$

By inserting this into (1.24) and using (1.26), we get that $-\Delta v(x^0) > 0$, but this contradicts with (1.25). This completes the proof. In the case that U is unbounded, the same arguments apply since the additional assumption (1.23) guarantees that the minimum of v does not leak away to infinity. □

Remark 1.12. *As illustrated in the proof, conditions* (1.20) *and* (1.22) *are required only at the points where v attains its minimum or at points where u is negative.*

In our application of the above theorem, we will consider two cases:

(a) U is a "narrow" region,

(b) the coefficient $c(x)$ has sufficient decay at infinity.

First, we examine when U is a narrow region; namely, let us consider the narrow strip with width $\ell > 0$, i.e.,

$$U = \{x \in \mathbb{R}^n \,|\, 0 < x_1 < \ell\}.$$

We can take $\varphi(x) = \sin((x_1 + \epsilon)/\ell)$ so that $-\Delta\varphi = (1/\ell)^2\varphi$. Thus, $\lambda(x) = -(1/\ell)^2$, which can be "very negative" if ℓ is suitably small.

Corollary 1.5 (Narrow region)**.** *If u satisfies* (1.21) *with bounded function $c(x)$, the width ℓ of the region U is sufficiently small, $c(x)$ satisfies* (1.22)*, i.e., $c(x) > \lambda(x) = -1/\ell^2$, then*

$$u \geq 0 \ \ in \ \ U.$$

In the case of (b) with $n \geq 3$, we can choose a positive number $q < n-2$ and take $\phi(x) = |x|^{-q}$, then a simple calculation yields

$$-\Delta\phi = \frac{q(n-2-q)}{|x|^2}\phi := -\lambda(x)\phi.$$

Therefore, if $c(x)$ has sufficient decay, the previous theorem implies the following.

Corollary 1.6 (Decay at infinity). *Assume there exists $R > 0$ such that*

$$c(x) > -\frac{q(n-2-q)}{|x|^2}, \ \ \textit{for all } |x| > R. \tag{1.27}$$

Suppose that

$$\lim_{|x|\to\infty} u(x)|x|^q = 0.$$

Let U be a region contained in $B_R^C(0)$. If u satisfies (5.5) on $\bar{U}$, then

$$u(x) \geq 0 \ \ \textit{for all } x \in U.$$

Remark 1.13. *As pointed out in the last remark, one can see that condition (1.27) is only required at points where u is negative.*

The following is a maximum principle for small volume domains due to Varadhan.

Theorem 1.19. *Suppose that U is an open bounded domain in $\mathbb{R}^n$ and $u \in C^2(U) \cap C^(\bar{U})$ satisfies*

$$\begin{cases} Lu \leq 0 & \textit{in } U, \\ u \leq 0 & \textit{on } \partial U. \end{cases}$$

Then there exists a $\delta > 0$, depending only on n, λ, Λ, $diam(U)$ and $\|c^-\|_{L^\infty(U)}$, such that if $|U| \leq \delta$, then $u \leq 0$ in U.

To prove Varadhan's "small volume" maximum principle requires a simplified case of the Alexandroff-Bakelman-Pucci (ABP) estimate, which we will study later in Chapter 4, e.g., for a close variant of the full ABP estimate, see Lemma 4.1. For convenience, we state here the simplified version in order to prove Varadhan's maximum principle.

Lemma 1.5. *Suppose that U is an open bounded domain in $\mathbb{R}^n$, $f \in C(U)$ and $u \in C^2(U) \cap C^(\bar{U})$ satisfies*

$$\begin{cases} Lu \leq f & \textit{in } U, \\ u \leq 0 & \textit{on } \partial U. \end{cases}$$

Then there exists a positive constant $C(n, \lambda, \Lambda)$ depending only on n, λ, and Λ, such that

$$\max_U u^+ \leq C(n,\lambda,\Lambda) diam(U) \Big(\int_U |f^-|^n \, dx \Big)^{1/n}$$

Proof of Theorem 1.19. If $c \geq 0$ in U, then the result just follows from the standard maximum principle. Hence, we can assume $c^- \not\equiv 0$ in U.

Now, on the contrary, let us suppose $u^+ \not\equiv 0$. Then u satisfies $u \leq 0$ on the boundary ∂U and

$$-\sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u + \sum_{i=1}^{n} b^i(x) D_i u + c^+(x) u \leq c^-(x) u \leq c^- u^+ \quad \text{in } U.$$

By Lemma 1.5, there exists a positive constant $C(n, \lambda, \Lambda)$ such that

$$\max_U u^+ \leq C(n, \lambda, \Lambda) diam(U) \|c^-\|_{L^\infty(U)} |U|^{1/n} \max_U u^+. \tag{1.28}$$

Setting

$$\delta < [C(n, \lambda, \Lambda) diam(U) \|c^-\|_{L^\infty(U)}]^{-n},$$

so that if $|U| < \delta$, then (1.28) implies that $\max_U u^+ \leq 0$, which is a contradiction. □

1.3 The Newtonian and Riesz Potentials

1.3.1 The Newtonian Potential and Green's Formula

Definition 1.2. *The function*

$$\Gamma(x) := \begin{cases} \dfrac{1}{2\pi} \log |x|, & \text{if } n = 2, \\ \dfrac{1}{\omega_n (n-2)} \dfrac{1}{|x|^{n-2}}, & \text{if } n \geq 3. \end{cases}$$

defined for all $x \in \mathbb{R}^n \backslash \{0\}$, is the fundamental solution of Laplace's equation. In addition, if $f \in L^p(U)$ for $1 < p < \infty$, then the Newtonian potential of f is defined by

$$w(x) := \int_{\mathbb{R}^n} \Gamma(x - y) f(y)\, dy.$$

The following theorem is a basic result which states that the kernel Γ in the Newtonian potential is the fundamental solution of Poisson's equation. In fact, it is a distribution solution for $-\Delta u = \delta_0$ in $\mathbb{R}^n$, where δ_{x_0} denotes the standard Dirac delta distribution supported at $x_0 \in \mathbb{R}^n$.

Theorem 1.20. *The function $\Gamma : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R}$ satisfies $-\Delta \Gamma = \delta_0$ in $\mathbb{R}^n$ in the distribution sense.*

Proof. Without loss of generality, we assume $n \geq 3$. Pick any $\varphi \in C_c^\infty(\mathbb{R}^n)$. For each fixed $\epsilon > 0$, straightforward calculations will reveal that

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_\epsilon(0)} \Gamma(x)(-\Delta\varphi)\, dx &= \int_{\mathbb{R}^n \setminus B_\epsilon(0)} -\Delta\Gamma(x)\varphi\, dx \\
&\quad + \frac{1}{(n-2)\omega_n}\int_{\partial B_\epsilon(0)} D(|x|^{2-n})\cdot\nu\varphi\, dS - \frac{1}{(n-2)\omega_n}\int_{\partial B_\epsilon(0)} |x|^{2-n}\frac{\partial\varphi}{\partial\nu}\, dS \\
&= \frac{1}{(n-2)\omega_n}\int_{\partial B_\epsilon(0)} (n-2)|x|^{1-n}\cdot\nu\varphi\, dS - C\|D\varphi\|_{L^\infty(\mathbb{R}^n)} o(1) \\
&= \frac{1}{|\partial B_\epsilon(0)|}\int_{\partial B_\epsilon(0)} \varphi\, dS + o(1) \longrightarrow \varphi(0) \quad \text{as } \epsilon \longrightarrow 0^+.
\end{aligned}$$

Here we used the fact that $\Delta\Gamma(x) = 0$ in $\mathbb{R}^n \setminus B_\epsilon(0)$, and

$$\lim_{\epsilon\to 0^+} \frac{1}{|\partial B_\epsilon(0)|}\int_{\partial B_\epsilon(0)} \varphi\, dS \longrightarrow \varphi(0)$$

by Lebesgue's differentiation theorem, Theorem 3.4. $\square$

The Newtonian potential also provides a solution to the Poisson equation on bounded and unbounded domains.

Theorem 1.21. *Let $f \in C_c^2(\mathbb{R}^n)$ and define u to be the Newtonian potential of f. Then*

(i) $u \in C^2(\mathbb{R}^n)$,

(ii) $-\Delta u = f$ *in* $\mathbb{R}^n$.

Proof. **Step 1:** Clearly,

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y)f(y)\, dy = \int_{\mathbb{R}^n} \Gamma(y)f(x-y)\, dy,$$

therefore,

$$\frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} \Gamma(y)\Big(\frac{f(x+he_i-y)-f(x-y)}{h}\Big)\, dy,$$

where $h \neq 0$ and $e_i = (0,\ldots,1,0,\ldots 0)$ where the 1 is in the i^{th} slot. Of course,

$$\frac{f(x+he_i-y)-f(x-y)}{h} \longrightarrow f_{x_i}(x-y) \;\text{ uniformly on } \mathbb{R}^n \;\text{ as }\; h \longrightarrow 0,$$

and thus for $i = 1, 2, \ldots, n$,

$$u_{x_i}(x) = \int_{\mathbb{R}^n} \Gamma(y) f_{x_i}(x-y)\, dy.$$

Likewise, for $i = 1, 2, \dots, n$,

$$u_{x_i x_j}(x) = \int_{\mathbb{R}^n} \Gamma(y) f_{x_i x_j}(x-y)\, dy$$

and this shows u is C^2 since the right-hand side of the last identity is continuous.

Step 2: Fix $\varepsilon > 0$ and suppose $n \geq 3$. Due to the singularity of fundamental solution at the origin, we must be careful in our calculation. Namely, we first consider the splitting

$$\Delta u(x) = \int_{B_\varepsilon(0)} \Gamma(y) \Delta_x f(x-y)\, dy + \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(y) \Delta_x f(x-y)\, dy := I_\varepsilon^1 + I_\varepsilon^2. \tag{1.29}$$

Then, polar coordinates implies

$$|I_\varepsilon^1| \leq C \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \int_{B_\varepsilon(0)} |\Gamma(y)|\, dy \leq C\varepsilon^{n-(n-2)} \leq C\varepsilon^2. \tag{1.30}$$

Integration by parts implies

$$\begin{aligned} I_\varepsilon^2 &= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(y) \Delta_y f(x-y)\, dy \\ &= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} D\Gamma(y) \cdot D_y f(x-y)\, dy + \int_{\partial B_\varepsilon(0)} \Gamma(y) \frac{\partial f}{\partial \nu}(x-y)\, dS(y) \\ &:= J_\varepsilon^1 + J_\varepsilon^2, \end{aligned} \tag{1.31}$$

where ν denotes the inward pointing unit normal along $\partial B_\varepsilon(0)$. Now,

$$|J_\varepsilon^2| \leq \|Df\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B_\varepsilon(0)} |\Gamma(y)|\, dS(y) \leq C\varepsilon. \tag{1.32}$$

Again, integration by parts and since Γ is harmonic away from the origin, we get

$$\begin{aligned} J_\varepsilon^1 &= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Delta\Gamma(y) f(x-y)\, dy - \int_{\partial B_\varepsilon(0)} \frac{\partial \Gamma}{\partial \nu}(y) f(x-y)\, dS(y) \\ &= -\int_{\partial B_\varepsilon(0)} \frac{\partial \Gamma}{\partial \nu}(y) f(x-y)\, dS(y). \end{aligned} \tag{1.33}$$

Now, it is clear that $D\Gamma(y) = -\frac{1}{\omega_n} \frac{y}{|y|^n}$ $(y \neq 0)$ and $\nu = -y/|y| = -y/\varepsilon$ on $\partial B_\varepsilon(0)$. Thus,

$$\frac{\partial \Gamma}{\partial \nu}(y) = \nu \cdot D\Gamma(y) = \frac{1}{\omega_n \varepsilon^{n-1}} \text{ on } \partial B_\varepsilon(0).$$

Hence,

$$J_\varepsilon^1 = -\frac{1}{\omega_n \varepsilon^{n-1}} \int_{\partial B_\varepsilon(0)} f(x-y)\, dS(y) = -\frac{1}{|B_\varepsilon(0)|} \int_{\partial B_\varepsilon(0)} f(x-y)\, dS(y) \longrightarrow -f(x) \tag{1.34}$$

as $\varepsilon \longrightarrow 0$. Hence, combining the estimates (1.30)–(1.34) and sending $\varepsilon \longrightarrow 0$ in (1.29), we obtain $-\Delta u(x) = f(x)$ and this completes the proof. □

Remark 1.14. *The proof above remains valid in the case where* $n = 2$ *except that the estimates for* I_ε^1 *and* J_ε^2 *become*

$$|I_\varepsilon^1| \leq C\varepsilon^2 |\log \varepsilon| \;\; \text{and} \;\; |J_\varepsilon^2| \leq C\varepsilon |\log \varepsilon|.$$

The previous result can be easily refined to include both bounded and unbounded open domains whose sources are locally Hölder continuous. We state the result below but omit the proof, which the reader can find in Gilbarg and Trudinger [13] (see Lemma 4.2).

Theorem 1.22. *Let* f *be bounded and locally Hölder continuous with exponent* $\alpha \in (0, 1]$ *in an open domain* $U \subset \mathbb{R}^n$*, and define* u *to be the Newtonian potential of* f*. Then*

(i) $u \in C^2(U)$*,*

(ii) $-\Delta u = f$ *in* U*, and*

(iii) for any $x \in U$*,*

$$D_{ij}u(x) = \int_{U_0} D_{ij}\Gamma(x-y)(f(x)-f(y))\, dy - f(x) \int_{\partial U_0} D_i\Gamma(x-y)\nu_j(y)\, dS(y), \; i, j = 1, 2, \ldots, n;$$

where U_0 *is any domain containing* U *for which the divergence theorem holds and* f *is extended to vanish outside* U*.*

In fact, the Newtonian potential for such source terms f is the unique solution of Poisson's equation in $\mathbb{R}^n$ modulo constants.

Theorem 1.23. *Let* $n \geq 3$*,* $f \in C_c^2(\mathbb{R}^n)$ *and suppose* $u \in C^2(\mathbb{R}^n)$ *is a bounded solution of* $-\Delta u = f$ *in* $\mathbb{R}^n$*. Then*

$$u(x) = \frac{1}{\omega_n(n-2)} \int_{\mathbb{R}^n} |x-y|^{2-n} f(y)\, dy + C$$

for some constant C*.*

Proof. Define $w = \Gamma * f$ and set $v = u - w$. Obviously, v is bounded since both u and w are bounded and v is harmonic in $\mathbb{R}^n$. Hence, v is constant, i.e., $u = w + C$ for some constant C thanks to the Liouville theorem. □

1.3.2 Riesz Potentials and Sharp Hardy-Littlewood-Sobolev Inequalities

From the previous theorem, we see that the Newtonian potential provides an explicit formula for solutions of Poisson's equation. On the other hand, the integral equation provides a simple example of a singular integral operator, which can be naturally extended to more general singular integral operators such as the Riesz potential. Remarkably yet not surprisingly, the Riesz potentials are very closely related to problems involving fractional Laplacians such as the Lane-Emden and Hardy-Littlewood-Sobolev systems. We give a definition of Riesz potentials here and briefly discuss their boundedness in L^p spaces.

Definition 1.3. *Let* α *be a complex number with positive real part* $Re\ \alpha > 0$*. The Riesz potential of order* α *is the operator*

$$I_\alpha = (-\Delta)^{-\alpha/2}.$$

In particular,

$$I_\alpha(f)(x) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

where $C_{n,\alpha} = 2^{-\alpha}\pi^{-\frac{n}{2}}\frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}$ *and the integral is convergent if* $f \in \mathcal{S}$*, i.e.,* f *belongs in the Schwartz class.*

The following result is the well-known Hardy-Littlewood-Sobolev (HLS) inequality, which indicates when boundedness holds for the Riesz potentials in Lebesgue spaces.

Theorem 1.24 (Hardy-Littlewood-Sobolev inequality)**.** *Let* $0 < \alpha < n$ *and* $p, r > 1$ *such that*

$$\frac{1}{p} + \frac{1}{r} + \frac{n-\alpha}{n} = 2.$$

Then

$$\Big| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^{n-\alpha}}\, dxdy\Big| \le C_{n,p,\alpha}\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^r(\mathbb{R}^n)} \tag{1.35}$$

for any $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^r(\mathbb{R}^n)$ *where* $C_{n,p,\alpha}$ *is a positive constant.*

Remark 1.15. *The sharp constant in the HLS inequality satisfies*

$$C_{n,p,\alpha} \le \frac{n}{\alpha}\frac{1}{pr}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{1-\alpha/n}\left(\left(\frac{1-\alpha/n}{1-1/p}\right)^{1-\alpha/n} + \left(\frac{1-\alpha/n}{1-1/r}\right)^{1-\alpha/n}\right),$$

In particular, if $p = r = 2n/(n+\alpha)$*, then*

$$C_{n,p,\alpha} = C(n,\alpha) = \pi^{(n-\alpha)/2}\frac{\alpha}{n+\alpha}\left(\frac{\Gamma(n)}{\Gamma(n/2)}\right)^{\alpha/n}.$$

In this case, equality in (1.35) *holds if and only if* $g \equiv cf$ *and*

$$f(x) = A(\gamma^2 + |x-x_0|^2)^{-(n+\alpha)/2}$$

for some constant A*,* $\gamma \in \mathbb{R}\backslash\{0\}$*, and some point* $x_0 \in \mathbb{R}^n$*. We do not have the necessary tools to establish the sharp HLS inequality at this time, however, we shall do this in Section 7.5 of Chapter 7.*

The following is an equivalent formulation of the HLS inequality. It determines the conditions on the exponents p and q that guarantee $I_\alpha : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n)$ is a bounded linear operator.

Theorem 1.25. *Let $\alpha \in (0,n)$, $1 < p < q < \infty$, $f \in L^p(\mathbb{R}^n)$ and*

$$\frac{n}{n-\alpha} < q \ \ \textit{with} \ \ \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}, \ \textit{i.e.,} \ \ p = \frac{nq}{n+\alpha q}.$$

Then

$$\|I_\alpha(f)\|_{L^q(\mathbb{R}^n)} \leq C_{n,p,\alpha}\|f\|_{L^p(\mathbb{R}^n)}.$$

For completeness, we shall give a proof of this version of the HLS inequality in Section 3.1.5 after developing the necessary tools. We prove Theorem 1.24 as a consequence.

Proof of Theorem 1.24. The proof is just by duality; that is, it follows easily from Hölder's inequality and Theorem 1.25. Indeed, we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x-y|^{n-\alpha}}\,dxdy\Big| &\leq \|fI_\alpha(g)\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}\|I_\alpha g\|_{L^q(\mathbb{R}^n)} \\ &\leq C_{n,p,\alpha}\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^r(\mathbb{R}^n)},\end{aligned}$$

where $q = p/(p-1)$ and we applied the HLS inequality, $\|I_\alpha g\|_{L^{p/(p-1)}(\mathbb{R}^n)} \leq C_{n,p,\alpha}\|g\|_{L^r(\mathbb{R}^n)}$, since

$$\frac{1}{p} + \frac{1}{r} + \frac{n-\alpha}{n} = 2 \implies \frac{1}{r} - \frac{1}{q} = \frac{1}{r} - \frac{p-1}{p} = \frac{\alpha}{n}.$$

□

Remark 1.16. *Conversely, we can show that the boundedness of I_α implies the HLS inequality by another duality argument. Namely, by taking $r' = r/(r-1)$ and because $1/p + 1/r + (n-\alpha)/n = 2 \iff 1/p - 1/r' = \alpha/n$, Theorem 1.25 implies*

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x-y|^{n-\alpha}}\,dxdy\Big| &\leq |\langle I_\alpha f, g\rangle| \leq \|I_\alpha f\|_{L^{r'}(\mathbb{R}^n)}\|g\|_{L^r(\mathbb{R}^n)} \\ &\leq C_{n,p,\alpha}\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^r(\mathbb{R}^n)}.\end{aligned}$$

One interesting motivation for considering Riesz potentials is due to their close relationship with poly-harmonic equations. For instance, consider the system

$$\begin{cases} (-\Delta)^{\alpha/2}u = |x|^{\sigma_1}v^q, & u > 0, \quad \text{in } \mathbb{R}^n, \\ (-\Delta)^{\alpha/2}v = |x|^{\sigma_2}u^p, & v > 0, \quad \text{in } \mathbb{R}^n. \end{cases} \tag{1.36}$$

When $\alpha \in (0,n)$ is an even integer and $\sigma_1, \sigma_2 \in (-\alpha, \infty)$, (1.36) is equivalent to the integral system of Riesz potentials

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n}\frac{|y|^{\sigma_1}v(y)^q}{|x-y|^{n-\alpha}}dy, \ u > 0 \ \text{ in } \mathbb{R}^n, \\ v(x) = \displaystyle\int_{\mathbb{R}^n}\frac{|y|^{\sigma_2}u(y)^p}{|x-y|^{n-\alpha}}dy, \ v > 0 \ \text{ in } \mathbb{R}^n, \end{cases} \tag{1.37}$$

in the sense that a classical solution of one system, multiplied by a suitable constant if necessary, is also a solution of the other when $p, q > 1$, and vice versa. Interestingly, when $\sigma_i = 0$, the integral equations in (1.37) are the Euler–Lagrange equations of a functional under a constraint in the context of the HLS inequality. In particular, the extremal functions for obtaining the sharp constant in the HLS inequality are solutions of the system of integral equations. For more on the analysis of systems (1.36) and (1.37), we refer the reader to the papers [18, 19, 32, 33, 34, 35] and the references therein.

1.3.3 Green's Function and Representation Formulas of Solutions

Let $U \subset \mathbb{R}^n$ be an open and bounded subset with C^1 boundary ∂U. Our goal here is to find a representation of the solution of Poisson's equation

$$-\Delta u = f \ \text{ in } \ U$$

subject to the prescribed boundary condition

$$u = g \ \text{ on } \ \partial U.$$

We derive the formula for the Green's function to this problem. Fix $x \in U$ and choose $\epsilon > 0$ suitably small so that $B_\epsilon(x) \subset U$. Then, apply Green's formula on the region $V_\epsilon = U \backslash B_\epsilon(x)$ to $u(y)$ and $\Gamma(y-x)$ to get

$$\int_{V_\epsilon} u(y)\Delta\Gamma(y-x) - \Gamma(y-x)\Delta u(y)\, dy = \int_{\partial V_\epsilon} u(y)\frac{\partial \Gamma}{\partial \nu}(y-x) - \Gamma(y-x)\frac{\partial u}{\partial \nu}(y)\, dS(y). \tag{1.38}$$

Notice that $\Delta\Gamma(x-y) = 0$ for $x \neq y$ and that

$$\Big| \int_{\partial B_\epsilon(x)} \Gamma(y-x)\frac{\partial u}{\partial \nu}(y)\, dS(y)\Big| \leq C\epsilon^{n-1} \max_{\partial B_\epsilon(0)} |\Gamma| = o(1).$$

Then, similar to the proof of Theorem 1.21, we can show that

$$\int_{\partial B_\epsilon(x)} u(y)\frac{\partial \Gamma}{\partial \nu}(y-x)\, dS(y) = \frac{1}{|\partial B_\epsilon(x)|}\int_{\partial B_\epsilon(x)} u(y)\, dS(y) \longrightarrow u(x)$$

as $\epsilon \longrightarrow 0$. Hence, sending $\epsilon \longrightarrow 0$ in (1.38) yields

$$u(x) = \int_{\partial U} \Big\{ \Gamma(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Gamma}{\partial \nu}(y-x) \Big\}\, dS(y) - \int_U \Gamma(y-x)\Delta u(y)\, dy. \tag{1.39}$$

Indeed, identity (1.39) holds for any point $x \in U$ and any function $u \in C^2(U)$. This representation of u is almost complete since we know u satisfies Poisson's equation and its values on the boundary are given, i.e., we know the values of Δu in U and $u = g$ on ∂U. However, we do not know a priori the value of $\partial u/\partial \nu$ on ∂U. To circumvent this, we introduce, for fixed $x \in U$, a corrector function $\phi^x = \phi^x(y)$, solving the boundary-value problem

$$\begin{cases} \Delta\phi^x = 0 & \text{in } U, \\ \phi^x = \Gamma(y-x) & \text{on } \partial U. \end{cases} \tag{1.40}$$

As before, if we apply Green's formula once more, we obtain

$$\begin{aligned} -\int_U \phi^x(y)\Delta u(y)\,dy &= \int_{\partial U} u(y)\frac{\partial\phi^x}{\partial\nu}(y) - \phi^x(y)\frac{\partial u}{\partial\nu}(y)\,dS(y) \\ &= \int_{\partial U} u(y)\frac{\partial\phi^x}{\partial\nu}(y) - \Gamma(y-x)\frac{\partial u}{\partial\nu}(y)\,dS(y). \end{aligned} \tag{1.41}$$

Now introduce the Green's function for the region U.

Definition 1.4. *The Green's function for the region U is*

$$G(x,y) := \Gamma(y-x) - \phi^x(y) \ \ \textit{for} \ \ x,y \in U,\ x \neq y.$$

In view of this definition, adding (1.41) to (1.39) yields

$$u(x) = -\int_{\partial U} u(y)\frac{\partial G}{\partial\nu}(x,y)\,dS(y) - \int_U G(x,y)\Delta u(y)\,dy \quad (x \in U), \tag{1.42}$$

where

$$\frac{\partial G}{\partial\nu}(x,y) = D_y G(x,y)\cdot\nu(y)$$

is the outer normal derivative of G with respect to the variable y. Here, observe that the term $\partial u/\partial\nu$ no longer appears in identity (1.42).

In summary, suppose that $u \in C^2(\bar{U})$ is a solution of the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases} \tag{1.43}$$

for given continuous functions f and g. Then, we have basically shown the following.

Theorem 1.26 (Representation formula via Green's function)**.** *If $u \in C^2(\bar{U})$ solves problem* (1.43)*, then*

$$u(x) = -\int_{\partial U} g(y)\frac{\partial G}{\partial\nu}(x,y)\,dS(y) + \int_U G(x,y)f(y)\,dy \quad (x \in U). \tag{1.44}$$

If the geometry of U is simple enough, then we can actually compute the corrector function explicitly to obtain G. Two such examples are when U is the unit ball or the hyperbolic or half-space in $\mathbb{R}^n$.

1.3.4 Green's Function for a Half-Space

Consider the half-space

$$\mathbb{R}^n_+ = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \,|\, x_n > 0\},$$

whose boundary is given by $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. Although the half-space is unbounded and the calculations in the previous section assumed U was bounded, we can still use the same ideas to find the Green's function for the half-space. In order to do so, we adopt a reflection argument. Namely, if $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n_+$, we let $\tilde{x} = (x_1, x_2, \dots, -x_n)$, the reflection of x in the plane $\partial\mathbb{R}^n_+$. Then set

$$\phi^x(y) = \Gamma(y - \tilde{x}) = \Gamma(y_1 - x_1, \dots, y_{n-1} - x_{n-1}, y_n + x_n) \text{ for } x, y \in \mathbb{R}^n_+.$$

The idea is that this corrector ϕ^x is built from Γ by reflecting the singularity from $x \in \mathbb{R}^n_+$ to $\tilde{x} \notin \mathbb{R}^n_+$. Observe that

$$\phi^x(y) = \Gamma(y - x) \text{ if } y \in \partial\mathbb{R}^n_+,$$

and thus

$$\begin{cases} \Delta\phi^x = 0 & \text{in } \mathbb{R}^n_+, \\ \phi^x = \Gamma(y - x) & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{1.45}$$

as required. That is, we have the following definition.

Definition 1.5. *The Green's function for the half-space $\mathbb{R}^n_+$ is*

$$G(x, y) := \Gamma(y - x) - \Gamma(y - \tilde{x}) \text{ for } x, y \in \mathbb{R}^n_+,\ x \neq y.$$

Then

$$G_{y_n}(x, y) = \Gamma_{y_n}(y - x) - \Gamma_{y_n}(y - \tilde{x}) = \frac{-1}{\omega_n}\Big[\frac{y_n - x_n}{|y - x|^n} - \frac{y_n + x_n}{|y - \tilde{x}|^n}\Big].$$

Consequently, if $y \in \partial\mathbb{R}^n_+$,

$$\frac{\partial G}{\partial \nu}(x, y) = -G_{y_n}(x, y) = -\frac{2x_n}{\omega_n}\frac{1}{|x - y|^n}.$$

Now if u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ u = g & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{1.46}$$

then the representation formula (1.44) of the previous theorem suggests that

$$u(x) = \frac{2x_n}{\omega_n}\int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x - y|^n}\, dy \quad (x \in \mathbb{R}^n_+) \tag{1.47}$$

is the representation formula for the solution. Here, the function

$$K(x,y) := \frac{2x_n}{\omega_n}\frac{1}{|x-y|^n} \quad \text{for } x \in \mathbb{R}^n_+, y \in \partial\mathbb{R}^n_+$$

is called **Poisson's kernel** for $U = \mathbb{R}^n_+$ and (1.47) is called **Poisson's formula**. Now, let us prove that Poisson's formula indeed gives the formula for the solution of the boundary-value problem (1.46).

Theorem 1.27 (Poisson's formula for $\mathbb{R}^n_+$)**.** *Assume $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$, and define u by Poisson's formula* (1.47)*. Then*

(a) $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$,

(b) $\Delta u = 0$ in $\mathbb{R}^n_+$,

(c) $\lim\limits_{x\to x^0, x\in\mathbb{R}^n_+} u(x) = g(x^0)$ for each point $x^0 \in \partial\mathbb{R}^n_+$.

1.3.5 Green's Function for a Ball

If $U = B_1(0)$, we construct the Green's function through another reflection argument, but here we exploit an inversion through the unit sphere $\partial B_1(0)$.

Definition 1.6. *If $x \in \mathbb{R}^n\backslash\{0\}$, the point*

$$\tilde{x} = \frac{x}{|x|^2}$$

is called the point dual to x with respect to $\partial B_1(0)$. The mapping $x \mapsto \tilde{x}$ is inversion through the unit sphere $\partial B_1(0)$.

Obviously, the inversion maps points on the sphere to itself, maps the points in the ball to its exterior $\mathbb{R}^n\backslash B_1(0)$, and maps points in the exterior into the ball. Now fix $x \in B_1(0)$ and we want to find the corrector function $\phi^x = \phi^x(y)$ solving

$$\begin{cases} \Delta\phi^x = 0 & \text{in } B_1(0), \\ \phi^x = \Gamma(y-x) & \text{on } \partial B_1(0), \end{cases} \tag{1.48}$$

with the Green's function

$$G(x,y) = \Gamma(y-x) - \phi^x(y).$$

Notice that the mapping $y \mapsto \Gamma(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$. Thus $y \mapsto |x|^{2-n}\Gamma(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$. Hence,

$$\phi^x(y) := \Gamma(|x|(y-\tilde{x})) \tag{1.49}$$

is harmonic in $U = B_1(0)$. Furthermore, if $y \in \partial B_1(0)$ and $x \neq 0$,

$$|x|^2|y-\tilde{x}|^2 = |x|^2\Big(|y|^2 - 2\frac{y\cdot x}{|x|^2} + \frac{1}{|x|^2}\Big) = |x|^2 - 2y\cdot x + 1 = |x-y|^2.$$

That is, $|x-y|^{2-n} = (|x||y-\tilde{x}|)^{2-n}$ and so

$$\phi^x(y) = \Gamma(y-x) \quad (y \in \partial B_1(0)),$$

as required.

Definition 1.7. *The Green's function for the unit ball $B_1(0)$ is*

$$G(x,y) := \Gamma(y-x) - \Gamma(|x|(y-\tilde{x})) \quad (x,y \in B_1(0)). \tag{1.50}$$

Note that the same formula holds when $n = 2$, where the kernel Γ is of the logarithmic type. Now assume u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B_1(0), \\ u = g & \text{on } \partial B_1(0). \end{cases} \tag{1.51}$$

Then the representation formula (1.44) indicates that

$$u(x) = -\int_{\partial B_1(0)} g(y) \frac{\partial G}{\partial \nu}(x,y)\, dS(y). \tag{1.52}$$

Then, according to (1.50),

$$G_{y_i}(x,y) = \Gamma_{y_i}(y-x) - \Gamma(|x|(y-\tilde{x}))_{y_i}.$$

We calculate that

$$\Gamma_{y_i}(y-x) = \frac{1}{\omega_n} \frac{x_i - y_i}{|x-y|^n},$$

and

$$\Gamma(|x|(y-\tilde{x}))_{y_i} = -\frac{1}{\omega_n} \frac{y_i|x|^2 - x_i}{(|x||y-\tilde{x}|)^n} = -\frac{1}{\omega_n} \frac{y_i|x|^2 - x_i}{|x-y|^n}$$

if $y \in \partial B_1(0)$. Then,

$$\frac{\partial G}{\partial \nu}(x,y) = \sum_{i=1}^n y_i G_{y_i}(x,y) = -\frac{1}{\omega_n} \frac{1}{|x-y|^n} \sum_{i=1}^n y_i((y_i - x_i) - y_i|x|^2 + x_i) = -\frac{1}{\omega_n} \frac{1-|x|^2}{|x-y|^n}.$$

Inserting this into (1.52) yields the representation formula

$$u(x) = \frac{1-|x|^2}{\omega_n} \int_{\partial B_1(0)} \frac{g(y)}{|x-y|^n}\, dS(y).$$

Actually, we can use a dilation argument to get the Green's function for $U = B_R(0)$. Namely, suppose now that u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B_R(0), \\ u = g & \text{on } \partial B_R(0). \end{cases} \tag{1.53}$$

It is easy to check that $\tilde{u}(x) = u(Rx)$ solves (1.51) with $\tilde{g} = g(Rx)$ replacing g. A simple change of variables yields **Poisson's formula**

$$u(x) = \frac{R^2 - |x|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{g(y)}{|x-y|^n} \, dS(y) \quad (x \in B_R(0)), \tag{1.54}$$

where the function

$$K(x,y) := \frac{R^2 - |x|^2}{\omega_n R} \frac{1}{|x-y|^n} \quad (x \in B_R(0), y \in \partial B_R(0))$$

is **Poisson's kernel** for the ball $U = B_R(0)$.

We have established Poisson's formula (1.55) under the assumption that a smooth solution of (1.53) exists. Indeed, the following theorem asserts that this formula does indeed give a solution.

Theorem 1.28 (Poisson's formula for the ball $B_R(0)$)**.** *Assume $g \in C(\partial B_R(0))$ and define u by Poisson's formula* (1.55)*. Then*

(a) $u \in C^\infty(B_R(0))$,

(b) $\Delta u = 0$ *in* $B_R(0)$,

(c) $\lim_{x \to x^0, x \in B_R(0)} u(x) = g(x^0)$ *for each point* $x^0 \in \partial B_R(0)$.

Observe that Harnack's inequality can be established directly from Poisson's formula (1.55).

Theorem 1.29 (Harnack's inequality)**.** *Suppose u is a non-negative harmonic function in $B_R(x_0)$. Then*

$$\Big(\frac{R}{R+r}\Big)^{n-2} \frac{R-r}{R+r} u(x_0) \le u(x) \le \Big(\frac{R}{R-r}\Big)^{n-2} \frac{R+r}{R-r} u(x_0)$$

where $r = |x - x_0| < R$.

Proof. By the regularity and translation invariance properties of harmonic functions, we may assume $x_0 = 0$ and $u \in C(\bar{B}_R)$. Thus, from Poisson's formula,

$$u(x) = \frac{R^2 - |x|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{u(y)}{|x-y|^n} \, dS(y) \quad (x \in B_R(0)). \tag{1.55}$$

Now, since $R - |x| \le |x - y| \le R + |x|$ for $|y| = R$, we obtain

$$\frac{1}{\omega_n R} \frac{R - |x|}{R + |x|} \Big(\frac{1}{R+|x|}\Big)^{n-2} \int_{\partial B_R} u(y) \, dS \le u(x) \le \frac{1}{\omega_n R} \frac{R + |x|}{R - |x|} \Big(\frac{1}{R-|x|}\Big)^{n-2} \int_{\partial B_R} u(y) \, dS.$$

In view of the mean value property,

$$u(0) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R} u(y) \, dS,$$

we insert this into the previous estimates to arrive at the desired result. $\square$

From this, we deduce the Liouville theorem.

Corollary 1.7. *If u is an entire function, i.e., it is harmonic in $U = \mathbb{R}^n$, and u is either bounded above or below, then u is necessarily constant.*

Proof. By shifting, we may assume u is non-negative in $\mathbb{R}^n$. Then take any point $x \in \mathbb{R}^n$ and apply the previous Harnack's inequality to u on any ball $B_R(0)$ with $|x| < R$ to get

$$\Big(\frac{R}{R+|x|}\Big)^{n-2}\frac{R-|x|}{R+|x|}u(0) \leq u(x) \leq \Big(\frac{R}{R-|x|}\Big)^{n-2}\frac{R+|x|}{R-|x|}u(0).$$

Sending $R \longrightarrow +\infty$ here yields $u(x) = u(0)$, and we conclude that u is constant everywhere in $\mathbb{R}^n$ since x was chosen arbitrarily. $\square$

1.4 Hölder Regularity for Poisson's Equation

Let us motivate the consideration of Hölder spaces $C^{k,\alpha}$ rather than the classical C^k spaces when dealing with regularity and solvability of elliptic problems of the form $Lu = f$ in U.

For instance, if $f \in C_0^\infty(U)$ and $\Gamma = \Gamma(x)$ is the fundamental solution of Laplace's equation, then the Newtonian potential of f, i.e., $w = \Gamma * f$ or

$$w(x) = \int_U \Gamma(x-y)f(y)\,dy,$$

belongs to $C^\infty(\bar{U})$. However, if f is merely just continuous, then w is not necessarily twice differentiable.

Generally, $Lu = f$ in U is uniquely solvable for all $f \in C^2(U)$ in that there exists a unique solution $u \in C^2(U)$ for each such f; namely, the elliptic operator $L : C^2(U) \to C^2(U)$ is a bijective mapping. On the other hand, we naturally ask if for every $f \in C(U)$ the equation $Lu = f$ has a solution u in $C^2(U)$. Interestingly enough, this is not true and so the mapping $L : C^2(U) \to C(U)$ is not bijective. For instance, if $L = -\Delta$ or $L = -(\Delta - 1)$ and for the equation $Lu = f$, it is not true that for every $f \in C(U)$ the corresponding solution u belongs in $C^2(U)$ (see the example given below). Fortunately, if we hope to recover the bijectivity of the map L, we must instead consider the Hölder space $C^\alpha(U)$ in place of $C(U)$.

Remark 1.17. *One instance where the bijectivity (namely, the invertibility) of the map L becomes very important is in the method of continuity (see Section 2.6). This method makes use of the bijection of the solution map and the global $C^{2,\alpha}$ regularity estimates to prove existence results to general elliptic boundary value problems. Therefore, this gives further motivation and a glimpse of some topics examined in the later chapters.*

Example: Let us provide an example in which the solvability of $-\Delta u = f$ for a carefully chosen continuous f fails within the class of C^2 solutions. Take the continuous but not

Hölder continuous function

$$f(x) = \frac{x_1^2 - x_2^2}{2|x|^2}\Big(\frac{n+2}{(-\log|x|)^{1/2}} + \frac{1}{2(-\log|x|)^{3/2}}\Big),$$

set

$$g(x) = \sqrt{-\log R}(x_2^2 - x_1^2),$$

and let $U = B_R(0)$ with $R < 1$. Then

$$u(x) = (x_2^2 - x_1^2)(-\log|x|)^{1/2}$$

belongs to $C(\bar{B}_R(0)) \cap C^\infty(\bar{B}_R(0)\backslash\{0\})$ and satisfies

$$\begin{cases} -\Delta u = f & \text{in } B_R(0)\backslash\{0\}, \\ u = g & \text{on } \partial B_R(0), \end{cases} \tag{1.56}$$

but u is not in $C^2(B_R(0))$ since we can check that $\lim_{|x|\to 0} D_{11}u = -\infty$. To see this, assume there exists such a classical solution v. Then $w = u - v$ is harmonic in $B_R(0)\backslash\{0\}$, but basic theory on removable singularities of harmonic functions, see Theorem 1.5, ensures that w can be redefined at the origin so that w is harmonic in $B_R(0)$. Thus, w is $C^2(B_R(0))$ and therefore u must also belong to $C^2(B_R(0))$. Hence, $\lim_{|x|\to 0} D_{11}u$ exists and we arrive at a contradiction.

In view of the above observations, we should assume the data f is Hölder continuous. We define what this is among other related concepts in the next subsection.

1.4.1 The Hölder function spaces

We first introduce some definitions, particularly the Hölder functions and function spaces. Let x_0 be a point in $\mathbb{R}^n$ and f is a function defined on a bounded set U containing x_0.

Definition 1.8. *Let $\alpha \in (0,1)$. Then f is said to be* **Hölder continuous with exponent α at x_0** *if the quantity*

$$[f]_{\alpha;x_0} = \sup_U \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha}$$

is finite. Here $[f]_{\alpha;x_0}$ **is called the α-Hölder coefficient of f at x_0 with respect to U.**

Moreover, f is said to be **uniformly Hölder continuous with exponent α in U** *if the quantity*

$$[f]_{\alpha;U} = \sup_{x,y\in U,\, x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite.

Definition 1.9. *Likewise, f is said to be* **locally Hölder continuous with exponent α in U** *if f is uniformly Hölder continuous with exponent α on compact subsets of U. Obviously, the two notions of Hölder continuity coincide if U is a compact subset.*

Let $\alpha \in (0,1)$, $U \subset \mathbb{R}^n$ be an open set and k a non-negative integer.

Definition 1.10. *The* **Hölder spaces** $C^{k,\alpha}(\bar{U})$ *(respectively* $C^{k,\alpha}(U)$*) are defined as the subspaces of* $C^k(\bar{U})$ *(respectively* $C^k(U)$*) consisting of functions whose* k^{th} *order partial derivatives are uniformly Hölder continuous (respectively locally Hölder continuous) with exponent* α *in* U*. For short, we denote* $C^{0,\alpha}(\bar{U})$ *(respectively* $C^{0,\alpha}(U)$*) simply by* $C^\alpha(\bar{U})$ *(respectively* $C^\alpha(U)$*).*

Remark 1.18. *Let us discuss the endpoint cases for* α*. If* $\alpha = 1$, $C^\alpha(\bar{U})$ *(respectively* $C^\alpha(U)$*) is often called the space of uniformly Lipschitz continuous functions (respectively locally Lipschitz continuous functions). If* $\alpha = 0$, $C^{k,0}(\bar{U})$ *(respectively* $C^{k,0}(U)$*) are the usual* C^k *spaces. Moreover, for* $\alpha \in [0,1]$, $C_0^{k,\alpha}(U)$ *denotes the space of functions in* $C^{k,\alpha}(U)$ *having compact support in* U*.*

For $k = 0, 1, 2, \ldots$, consider the following seminorms

$$[u]_{k,0;U} = |D^k u|_{0;U} = \sup_{|\beta|=k} \sup_U |D^\beta u|,$$
$$[u]_{k,\alpha;U} = [D^k u]_{\alpha;U} = \sup_{|\beta|=k} [D^\beta u]_{\alpha,U}.$$

With these seminorms, we can define the norms

$$\|u\|_{C^k(\bar{U})} = |u|_{k;U} = |u|_{k,0;U} = \sum_{j=0}^{k} [u]_{j,0;U} = \sum_{j=0}^{k} |D^j u|_{0;U},$$
$$\|u\|_{C^{k,\alpha}(\bar{U})} = |u|_{k,\alpha;U} = |u|_{k;U} + [u]_{k,\alpha;U} = |u|_{k;U} + [D^k u]_{\alpha;U},$$

on the spaces $C^k(\bar{U})$, $C^{k,\alpha}(\bar{U})$. It is sometimes useful, especially in this section anyway, to consider non-dimensional norms on these spaces. In particular, if U is bounded with $d = diam(U)$, we set

$$\|u\|'_{C^k(\bar{U})} = |u|'_{k;U} = \sum_{j=0}^{k} d^j [u]_{j,0;U} = \sum_{j=0}^{k} d^j |D^j u|_{0;U},$$
$$\|u\|'_{C^{k,\alpha}(\bar{U})} = |u|'_{k,\alpha;U} = |u|'_{k;U} + d^{k+\alpha}[u]_{k,\alpha;U} = |u|'_{k;U} + d^{k+\alpha}[D^k u]_{\alpha;U}.$$

Not surprisingly, we have the following basic result, which we give without proof.

Theorem 1.30. *Let* $\alpha \in [0,1]$ *and* $U \subset \mathbb{R}^n$ *be an open domain. The spaces* $C^k(\bar{U})$, $C^{k,\alpha}(\bar{U})$ *equipped with the norms defined above are Banach spaces.*

The following algebra property holds: the product of Hölder continuous functions is again Hölder continuous. Namely, if $u \in C^\alpha(\bar{U})$, $v \in C^\beta(\bar{U})$, we have $uv \in C^\gamma(\bar{U})$ where $\gamma = \min\{\alpha, \beta\}$, and

$$\|uv\|_{C^\gamma(\bar{U})} \le \max(1, d^{\alpha+\beta-2\gamma}) \|u\|_{C^\alpha(\bar{U})} \|v\|_{C^\beta(\bar{U})},$$
$$\|uv\|'_{C^\gamma(\bar{U})} \le \|u\|'_{C^\alpha(\bar{U})} \|v\|'_{C^\beta(\bar{U})}.$$

1.4.2 The Dirichlet Problem for Poisson's Equation

We now develop the regularity properties of Newtonian potentials. We will use this to then show that Poisson's equation in a bounded domain U may be solved under the same boundary conditions for which Laplace's equation is solvable.

Lemma 1.6. *Let f be a bounded and integrable in U, and let w be the Newtonian potential of f. Then $w \in C^1(\mathbb{R}^n)$ and for any $x \in U$,*

$$D_i w(x) = \int_U D_i \Gamma(x-y) f(y)\, dy, \quad i = 1, 2, \ldots, n.$$

Proof. It is easy to check the following derivative estimates for Γ:

$$\begin{cases} |D_i \Gamma(x-y)| \leq \dfrac{1}{n\omega_n}|x-y|^{1-n}, \\ |D_{ij}\Gamma(x-y)| \leq \dfrac{1}{\omega_n}|x-y|^{-n}, \\ |D^\beta \Gamma(x-y) \leq C(n,|\beta|)|x-y|^{2-n-|\beta|}. \end{cases} \tag{1.57}$$

From this, the function

$$v(x) = \int_U D_i \Gamma(x-y) f(y)\, dy$$

is well-defined. We now show that $v = D_i w$. To do so, for $\epsilon > 0$, let $\eta_\epsilon(x,y) = \eta(|x-y|/\epsilon)$ where $\eta = \eta(|x|)$ is some non-negative radial function in $C^1(\mathbb{R})$ with *supp*$(\eta) \subseteq [0,1]$, *supp*$(\eta') \subseteq [0,2]$, and

$$\eta(|x|) := \begin{cases} 0, & \text{if } |x| \leq 1, \\ 1, & \text{if } |x| \geq 2. \end{cases}$$

Define

$$w_\epsilon(x) = \int_U \eta_\epsilon(x,y)\Gamma(x-y) f(y)\, dy,$$

which is obviously in $C^1(\mathbb{R}^n)$. Then, there holds,

$$v(x) - D_i w_\epsilon(x) = \int_{B_{2\epsilon}(x)} D_i \Big[(1-\eta_\epsilon(x,y))\Gamma(x-y)\Big] f(y)\, dy.$$

Hence, if $n \geq 3$,

$$|v(x) - D_i w_\epsilon(x)| \leq \|f\|_\infty \int_{B_{2\epsilon}(x)} |D_i\Gamma(x-y)| + \frac{2}{\epsilon}|\Gamma(x-y)|\, dy \leq \frac{2n\epsilon}{n-2}\|f\|_\infty.$$

Note that if $n = 2$, it follows that

$$|v(x) - D_i w_\epsilon(x)| \leq 4\epsilon(1 + |\ln 2\epsilon|).$$

In either case, we conclude that as $\epsilon \longrightarrow 0$, w_ϵ and $D_i w_\epsilon$ converge uniformly on compact subsets of $\mathbb{R}^n$ to w and v, respectively. Therefore, $w \in C^1(\mathbb{R}^n)$ and $v = D_i w$. $\square$

Lemma 1.7. *Let f be bounded and locally Hölder continuous in U with exponent $\alpha \in (0,1]$, and let w be the Newtonian potential of f. Then*

(a) $w \in C^2(U)$;

(b) $-\Delta w = f$ *in* U;

(c) For any $x \in U$,

$$D_{ij}w(x) = \int_{U_0} D_{ij}\Gamma(x-y)(f(y)-f(x))\,dy - f(x)\int_{\partial U_0} D_i\Gamma(x-y)\nu_j(y)\,dS_y, \quad i,j = 1,2,\ldots,n. \tag{1.58}$$

Here, U_0 is any domain containing U for which the divergence theorem holds and f is extended to vanish outside U.

Proof. Using the derivative estimates of (1.57) for $D^2\Gamma$ and since f is pointwise Hölder cotinuous in U, the function

$$u(x) = \int_{U_0} D_{ij}\Gamma(x-y)(f(y)-f(x))\,dy - f(x)\int_{\partial U_0} D_i\Gamma(x-y)\nu_j(y)\,dS_y,$$

is well-defined. Let $v = D_i w$ and define for $\epsilon > 0$,

$$v_\epsilon(x) = \int_U D_i\Gamma(x-y)\eta_\epsilon(x,y)f(y)\,dy,$$

where η_ϵ is the same test function as in the previous lemma. Obviously, $v_\epsilon \in C^1(U)$ and for $\epsilon > 0$ sufficiently small, differentiating leads to

$$\begin{aligned}
D_j v_\epsilon(x) &= \int_U D_j(D_i\Gamma(x-y)\eta_\epsilon(x,y))f(y)\,dy \\
&= \int_U D_j(D_i\Gamma(x-y)\eta_\epsilon(x,y))(f(y)-f(x))\,dy + f(x)\int_{U_0} D_j(D_i\Gamma(x-y)\eta_\epsilon(x,y))\,dy \\
&= \int_U D_j(D_i\Gamma(x-y)\eta_\epsilon(x,y))(f(y)-f(x))\,dy + f(x)\int_{\partial U_0} D_i\Gamma(x-y)\nu_j(y)\,dS_y.
\end{aligned}$$

Hence, by subtracting this from $u(x)$, we estimate that

$$\begin{aligned}
|u(x) - D_j v_\epsilon(x)| &= \Big|\int_{B_{2\epsilon}(x)} D_j[(1-\eta_\epsilon)D_i\Gamma(x-y)](f(y)-f(x))\,dy\Big| \\
&\le [f]_{\alpha;x}\int_{B_{2\epsilon}(x)} \Big(|D_{ij}\Gamma| + \frac{2}{\epsilon}|D_i\Gamma|\Big)|x-y|^\alpha\,dy \\
&\le \Big(\frac{n}{\alpha} + 4\Big)(2\epsilon)^\alpha [f]_{\alpha;x},
\end{aligned}$$

provided that $2\epsilon < dist(x, \partial U)$. Therefore, $D_j v_\epsilon$ converges to u uniformly on compact subsets of U as $\epsilon \longrightarrow 0$. Of course, v_ϵ converges to $v = D_i w$ as $\epsilon \longrightarrow 0$. Hence, $w \in C^2(U)$ and $u = D_{ij} w$. Then, if we set $U_0 = B_r(x)$ for r suitably large,

$$-\Delta w(x) = \frac{1}{\omega_n r^{n-1}} f(x) \int_{\partial B_r(x)} \nu_i(y)\nu_i(y)\, dS_y = f(x).$$

This completes the proof of the lemma. □

A consequence of Lemmas 1.6 and 1.7 is the following theorem. This result should be compared with Theorem 1.21 as it generalizes that result in that f is assumed to be bounded and locally Hölder continuous in U rather than the stronger condition that $f \in C_c^2(U)$.

Theorem 1.31. *Let U be a bounded domain and suppose that each point of ∂U is regular (with respect to the Laplacian). Then, if f is a bounded, locally Hölder continuous function in U, the classical Dirichlet problem*

$$\begin{cases} -\Delta u = f & \text{in } U, \\ \quad u = g & \text{on } \partial U, \end{cases} \tag{1.59}$$

is uniquely solvable for any continuous boundary values g in the class of classical solutions, i.e., $u \in C^2(U) \cap C(\bar{U})$.

Proof. Let w be the Newtonian potential of f and consider the function $v = u - w$. It is clear that $-\Delta v = 0$ in U and $v = g - w$ on ∂U, but it is obvious that the unique solvability of this boundary-value problem for Laplace's equation will imply the desired result. Now, the existence of classical solutions of Laplace's equation follows from several methods, e.g. the Perron method, which are provided in the next chapter, and the uniqueness of the solution is a consequence of the maximum principles. □

Remark 1.19. *Here, a boundary point will be called regular (with respect to the Laplacian) if there exists a barrier function at that point. For the definition of a barrier function, see* (2.22) *in the next chapter discussing Perron's method. There we shall see that if ∂U is C^2 then each point on the boundary is indeed regular. Furthermore, the regularity theory below indicates that the unique solution of the above Dirichlet problem on a Euclidean ball domain belongs to $C^{2,\alpha}(U) \cap C(\bar{U})$*

Remark 1.20. *If $U = B_R(0)$, the last theorem follows from the two preceding lemmas and Poisson's formula* (1.55) *for the ball. In fact, we even have an explicit representation of the unique solution, which is given by*

$$u(x) = \int_{\partial B_R(0)} K(x,y) g(y)\, dS_y + \int_{B_R(0)} G(x,y) f(y)\, dy,$$

where $K(x,y)$ and $G(x,y) = \Gamma(y-x) - \phi^x(y)$ are Poisson's kernel and the Green's function on the ball, respectively. In particular, for all $x, y \in B_R(0)$, $x \neq y$,

$$G(x,y) = \Gamma(y-x) - \Gamma\Big(\frac{|x|}{R}(y - \frac{R^2}{|x|^2}x)\Big). \tag{1.60}$$

1.4.3 Interior Hölder Estimates for Second Derivatives

For concentric balls of radius $R > 0$ centered at x_0 in $\mathbb{R}^n$, we set $B_1 = B_R(x_0)$ and $B_2 = B_{2R}(x_0)$.

Lemma 1.8. *Suppose that $f \in C^\alpha(\bar{B}_2)$, $\alpha \in (0,1)$, and let w be the Newtonian potential of f in B_2. Then $w \in C^{2,\alpha}(\bar{B}_1)$ and*

$$|D^2 w|'_{0,\alpha;B_1} \le C(n,\alpha)|f|'_{0,\alpha;B_2},$$
$$|D^2 w|_{0;B_1} + R^\alpha [D^2 w]_{\alpha;B_1} \le C(n,\alpha)(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).$$

Remark 1.21. *For general domains $U_1 \subset B_1(x_0)$ and $B_2(x_0) \subset U_2$, and $f \in C^\alpha(\bar{U}_2)$ and w is the Newtonian potential of f over U_2. Then the statement of Lemma 1.8 with U_i replacing $B_i(x_0)$, $i = 1, 2$, respectively, still remains true.*

Proof of Lemma 1.8. For any $x \in B_1$, identity (1.58) yields

$$D_{ij}w(x) = \int_{B_2} D_{ij}\Gamma(x-y)[f(y) - f(x)]\,dy - f(x)\int_{\partial B_2} D_i\Gamma(x-y)\nu_j(y)\,dS_y \tag{1.61}$$

and thus, by the derivative estimates in (1.57),

$$\begin{aligned} |D_{ij}w(x)| &\le \frac{|f(x)|}{n\omega_n}R^{1-n}\int_{\partial B_2} dS_y + \frac{[f]_{\alpha;x}}{\omega_n}\int_{B_2}|x-y|^{\alpha-n}\,dy \\ &= 2^{n-1}|f(x)| + \frac{n}{\alpha}(3R)^\alpha [f]_{\alpha;x} \le C(n,\alpha)(|f(x)| + R^\alpha [f]_{\alpha;x}). \end{aligned} \tag{1.62}$$

Then, again (1.58) implies that for any other point $\bar{x} \in B_1$ we have

$$D_{ij}w(\bar{x}) = \int_{B_2} D_{ij}\Gamma(\bar{x}-y)[f(y) - f(\bar{x})]\,dy - f(\bar{x})\int_{\partial B_2} D_i\Gamma(\bar{x}-y)\nu_j(y)\,dS_y. \tag{1.63}$$

Set $\delta = |x - \bar{x}|$ and $\xi = (x+\bar{x})/2$. Subtracting (1.63) from (1.61) yields

$$D_{ij}w(x) - D_{ij}w(\bar{x}) = f(x)I_1 + [f(x) - f(\bar{x})]I_2 + I_3 + I_4 + [f(x) - f(\bar{x})]I_5 + I_6,$$

where

$$\begin{aligned} I_1 &= \int_{\partial B_2}[D_i\Gamma(x-y) - D_i\Gamma(\bar{x}-y)]\nu_j(y)\,dS_y, \\ I_2 &= \int_{\partial B_2} D_i\Gamma(x-y)\nu_j(y)\,dS_y, \\ I_3 &= \int_{B_\delta(\xi)} D_{ij}\Gamma(x-y)[f(x) - f(y)]\,dy, \\ I_4 &= \int_{B_\delta(\xi)} D_{ij}\Gamma(\bar{x}-y)[f(y) - f(\bar{x})]\,dy, \\ I_5 &= \int_{B_2\setminus B_\delta(\xi)} D_{ij}\Gamma(x-y)\,dy, \\ I_6 &= \int_{B_2\setminus B_{\delta(\xi)}}[D_{ij}\Gamma(x-y) - D_{ij}\Gamma(\bar{x}-y)][f(\bar{x}) - f(y)]\,dy. \end{aligned}$$

We estimate each term I_i: For some $\tilde{x}$ between x and $\bar{x}$,

$$\begin{aligned}|I_1| &\le |x-\bar{x}| \int_{\partial B_2} |DD_i\Gamma(\tilde{x}-y)|\, dS_y \\ &\le \frac{n^2 2^{n-1}|x-\bar{x}|}{R} \text{ (since } |\tilde{x}-y| \ge R \text{ for } y \in \partial B_2) \\ &\le n^2 2^{n-\alpha}\Big(\frac{\delta}{R}\Big)^{\alpha} \text{ (since } \delta = |x-\bar{x}| < 2R),\end{aligned}$$

$$|I_2| \le \frac{1}{n\omega_n} R^{1-n} \int_{\partial B_2} dS_y = 2^{n-1},$$

and

$$\begin{aligned}|I_3| &\le \int_{B_\delta(\xi)} |D_{ij}\Gamma(x-y)||f(x)-f(y)|\, dy \\ &\le \frac{1}{\omega_n}[f]_{\alpha;x} \int_{B_{(3/2)\delta}(x)} |x-y|^{\alpha-n}\, dy \\ &\le \frac{n}{\alpha}\Big(\frac{3\delta}{2}\Big)^{\alpha}[f]_{\alpha;x}.\end{aligned}$$

Similarly,

$$|I_4| \le \frac{n}{\alpha}\Big(\frac{3\delta}{2}\Big)^{\alpha}[f]_{\alpha;\bar{x}},$$

$$\begin{aligned}|I_5| &= \Big| \int_{\partial(B_2\setminus B_\delta(\xi))} D_i\Gamma(x-y)\nu_j(y)\, dS_y \Big| \\ &\le \Big| \int_{\partial B_2} D_i\Gamma(x-y)\nu_j(y)\, dS_y \Big| + \Big| \int_{\partial B_\delta(\xi)} D_i\Gamma(x-y)\nu_j(y)\, dS_y \Big| \\ &\le 2^{n-1} + \frac{1}{n\omega_n}\Big(\frac{\delta}{2}\Big)^{1-n} \int_{\partial B_\delta(\xi)} dS_y = 2^{n-1} + 2^{n-1} = 2^n,\end{aligned}$$

$$\begin{aligned}|I_6| &\le |x-\bar{x}| \int_{B_2\setminus B_\delta(\xi)} |DD_{ij}\Gamma(\tilde{x}-y)||f(\bar{x})-f(y)|\, dy \text{ (for some } \tilde{x} \text{ between } x \text{ and } \bar{x}) \\ &\le c(n)\delta \int_{|y-\xi|\ge\delta} \frac{|f(\bar{x})-f(y)|}{|\tilde{x}-y|^{n+1}}\, dy \\ &\le c\delta[f]_{\alpha;\bar{x}} \int_{|y-\xi|\ge\delta} |\xi-y|^{\alpha-n-1}\, dy \text{ (since } |\bar{x}-y| \le (3/2)|\xi-y| \le 3|\tilde{x}-y|) \\ &\le c(n)(1-\alpha)^{-1}2^{n+1}\Big(\frac{3}{2}\Big)^{\alpha}\delta^{\alpha}[f]_{\alpha;\bar{x}}.\end{aligned}$$

Combining these estimates gives us

$$|D_{ij}w(\bar{x}) - D_{ij}w(x)| \le C(n,\alpha)\Big(R^{-\alpha}|f(x)| + [f]_{\alpha;x} + [f]_{\alpha;\bar{x}}\Big)|x-\bar{x}|^{\alpha}.$$

Hence, this along with (1.62) completes the proof of the lemma. □

Theorem 1.32. *Let $f \in C_0^{\alpha}(\mathbb{R}^n)$ and suppose $u \in C_0^2(\mathbb{R}^n)$ satisfy Poisson's equation,*

$$-\Delta u = f \ \text{ in } \mathbb{R}^n.$$

Then $u \in C_0^{2,\alpha}(\mathbb{R}^n)$, and if $B = B_R(x_0)$ is any ball containing the support of u, then

$$|D^2u|'_{0,\alpha;B} \le C(n,\alpha)|f|'_{0,\alpha;B} \quad \text{and} \quad |u|'_{1,B} \le C(n)R^2|f|_{0;B}.$$

Proof. As indicated in Theorem 1.21 or Lemma 1.7, we can conclude that $u = \Gamma * f$, even if it was assumed there that $f \in C_c^2(\mathbb{R}^n)$ as it still holds true even when $f \in C_0^{\alpha}(\mathbb{R}^n)$. The estimates for Du and D^2u follow, respectively, from Lemma 1.6 and Lemma 1.8 and the fact that f has compact support. The estimate for $|u|_{0;B}$ follows at once from that for Du. □

The restriction that u and f have compact support in the last theorem can be removed.

Theorem 1.33. *Let U be a domain in $\mathbb{R}^n$ and let $f \in C^{\alpha}(U)$, $\alpha \in (0,1)$, and let $u \in C^2(U)$ satisfy Poisson's equation, $-\Delta u = f$ in U. Then $u \in C^{2,\alpha}(U)$ and for any two concentric balls $B_R(x_0)$, $B_{2R}(x_0) \subset\subset U$, we have*

$$|u|'_{2,\alpha;B_R(x_0)} \le C(n,\alpha)(|u|_{0;B_{2R}(x_0)} + R^2|f|'_{0,\alpha;B_{2R}(x_0)}). \tag{1.64}$$

A consequence of the interior estimate (1.64) is the equicontinuity on compact subsets of the second derivatives of any bounded set of solutions of Poisson's equation. Therefore, the Arzelà–Ascoli theorem implies the following result on the compactness of solutions to Poisson's equation.

Corollary 1.8. *Any bounded sequence of solutions of Poisson's equation, $-\Delta u = f$ in U, where $f \in C^{\alpha}(U)$, contains a subsequence converging uniformly on compact subsets of U to another solution.*

As a consequence of this compactness result, we establish an existence result for the Dirichlet problem. Here, we denote $d_x = d_x(U) = dist(x, \partial U)$.

Theorem 1.34. *Let B be a ball in $\mathbb{R}^n$ and f be a function in $C^{\alpha}(B)$ for which*

$$\sup_{x \in B} d_x^{2-\beta}|f(x)| \le N < \infty$$

for some $\beta \in (0,1)$. Then there exists a unique function $u \in C^2(B) \cap C(\bar{B})$ satisfying

$$\begin{cases} -\Delta u = f & \text{in } B, \\ \quad\; u = 0 & \text{on } \partial B. \end{cases}$$

Furthermore, the solution u satisfies the estimate

$$\sup_{x\in B} d_x^{-\beta}|u(x)| \leq CN, \tag{1.65}$$

where $C = C(\beta)$.

Proof. Step 1: Estimate (1.65) follows from a simple barrier argument, i.e., let $B = B_R(x_0)$, $r = |x - x_0|$ and set $w(x) = (R^2 - r^2)^\beta$. A direct calculation will show that

$$\begin{aligned}\Delta w(x) = &-2\beta(R^2 - r^2)^{\beta-2}[n(R^2 - r^2) + 2(1-\beta)r^2] \\ \leq &-4\beta(1-\beta)R^2(R^2 - r^2)^{\beta-2} \leq -\beta(1-\beta)R^\beta(R-r)^{\beta-2}.\end{aligned}$$

Now suppose that $-\Delta u = f$ in B and $u = 0$ on ∂B. Since $d_x = R - r$, the hypothesis yields

$$|f(x)| \leq N d_x^{\beta-2} = N(R-r)^{\beta-2} \leq -C_0 N \Delta w,$$

where $C_0 = [\beta(1-\beta)R^\beta]^{-1}$. Hence,

$$-\Delta(C_0 N w \pm u) \geq 0 \text{ in } B, \text{ and } C_0 N w \pm u = 0 \text{ on } \partial B.$$

Therefore, the maximum principle implies

$$|u(x)| \leq C_0 N w(x) \leq C N d_x^\beta \quad \text{for } x \in B, \tag{1.66}$$

which implies (1.65) with constant $C = 2/\beta(1-\beta)$.

Step 2: We now prove the existence of u. Define

$$f_m = \begin{cases} m, & \text{if } f \geq m, \\ f, & \text{if } |f| \leq m, \\ -m, & \text{if } f \leq -m, \end{cases}$$

and let $\{B_k\}$ be a sequence of concentric balls exhausting B such that $|f| \leq k$ in B_k. We define u_m to be the solution of $-\Delta u_m = f_m$ in B and $u_m = 0$ on ∂B. By (1.65),

$$\sup_{x\in B} d_x^{-\beta}|u_m(x)| \leq C \sup_{x\in B} d_x^{2-\beta}|f_m(x)| \leq CN,$$

so that the sequence $\{u_m\}$ is uniformly bounded and $-\Delta u_m = f$ in B_k for $m \geq k$. Hence, by Corollary 1.8 applied successively to the sequence of balls B_k, we can extract a convergent subsequence of $\{u_m\}$ with limit point u in $C^2(B)$ satisfying $-\Delta u = f$ in B. Moreover, there holds $|u(x)| \leq C N d_x^\beta$ and so $u = 0$ on ∂B. This completes the proof of the theorem. □

1.4.4 Boundary Hölder Estimates for Second Derivatives

We may refine the interior Hölder regularity estimates by extending them up to the boundary. We focus only on ball domains but the results certainly apply to bounded and open domains with smooth boundary. We refer the reader to Chapter 3 for more details on obtaining regularity estimates up to the boundary for general smooth domains.

We start with some notation. Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \,|\, x_n > 0\}$ be the usual upper half-space with boundary $T = \partial\mathbb{R}^n_+$, $B_2 := B_{2R}(x_0)$, $B_1 = B_R(x_0)$ where $R > 0$ and $x_0 \in \mathbb{R}^n_+$. Moreover, set $B_2^+ := B_2 \cap \mathbb{R}^n_+$ and $B_1^+ = B_1 \cap \mathbb{R}^n_+$.

Lemma 1.9. *Let $f \in C^\alpha(\bar{B}_2^+)$ and let w be the Newtonian potential of f in B_2^+. Then $w \in C^{2,\alpha}(\bar{B}_1^+)$ and*

$$|D^2 w|'_{0,\alpha;B_1^+} \leq C|f|'_{0,\alpha;B_2^+} \tag{1.67}$$

where $C = C(n, \alpha)$.

Proof. We may assume B_2 intersects T, otherwise the result is already contained in Lemma 1.8. The representation (1.58) holds for $D_{ij}w$ within $U_0 = B_2^+$. If either i or $j \neq n$, then the portion of the boundary integral

$$\int_{\partial B_2^+} D_i\Gamma(x-y)\nu_j(y)\, dS_y = \int_{\partial B_2^+} D_j\Gamma(x-y)\nu_i(y)\, dS_y$$

on T vanishes since ν_i or ν_j equals to 0 there. The estimates in Lemma 1.8 for $D_{ij}w$ (i or $j \neq 0$) then proceed exactly as before with B_2 replaced with B_2^+, $B_\delta(\xi)$ replaced by $B_\delta(\xi) \cap B_2^+$ and ∂B_2 replaced by $\partial B_2^+ \backslash T$. Finally, $D_{nn}w$ can be estimated from the equation $-\Delta w = f$ and the estimates $D_{kk}w$ for $k = 1, 2, \ldots, n-1$. □

Theorem 1.35. *Let $u \in C^2(B_2^+) \cap C(\bar{B}_2^+)$, $f \in C^\alpha(\bar{B}_2^+)$, satisfy $-\Delta u = f$ in B_2^+, $u = 0$ on T. Then $u \in C^{2,\alpha}(\bar{B}_1^+)$ and we have*

$$|u|'_{2,\alpha;B_1^+} \leq C(|u|_{0;B_2^+} + R^2|f|'_{0,\alpha;B_2^+}) \tag{1.68}$$

where $C = C(n, \alpha)$.

Proof. Let $x' = (x_1, x_2, \ldots, x_{n-1})$, $x^* = (x', -x_n)$ and define

$$f^*(x) = f^*(x', x_n) := \begin{cases} f(x', x_n), & \text{if } x_n \geq 0, \\ f(x', -x_n), & \text{if } x_n \leq 0. \end{cases}$$

We assume that B_2 intersects T; otherwise Theorem 1.8 already implies estimate (1.68). Now set $B_2^- := \{x \in \mathbb{R}^n \,|\, x^* \in B_2^+\}$ and $D = B_2^+ \cup B_2^- \cup (B_2 \cap T)$. Then $f^* \in C^\alpha(\bar{D})$ and $|f^*|'_{0,\alpha;B_2^+}$.

Define

$$w(x) = \int_{B_2^+} [\Gamma(x-y) - \Gamma(x^*-y)] f(y)\,dy$$
$$= \int_{B_2^+} [\Gamma(x-y) - \Gamma(x-y^*)] f(y)\,dy, \tag{1.69}$$

so that $w(x',0) = 0$ and $-\Delta w = f$ in B_2^+. Observe that

$$\int_{B_2^+} \Gamma(x-y^*) f(y)\,dy = \int_{B_2^-} \Gamma(x-y) f^*(y)\,dy,$$

so then we get

$$w(x) = 2\int_{B_2^+} \Gamma(x-y) f(y)\,dy - \int_D \Gamma(x-y) f^*(y)\,dy.$$

Letting

$$w^*(x) = \int_D \Gamma(x-y) f^*(y)\,dy,$$

the remark following Lemma 1.8 with $U_1 = B_1^+$ and $U_2 = D$ implies that

$$|D^2 w^*|'_{0,\alpha;B_1^+} \le C|f^*|'_{0,\alpha;D} \le 2C|f|'_{0,\alpha;B_2^+}.$$

Combining this with Lemma 1.9 yields

$$|D^2 w|'_{0,\alpha;B_1^+} \le C|f|'_{0,\alpha;B_2^+}.$$

Now let $v = u - w$, then $\Delta v = 0$ in B_2^+ and $v = 0$ on T. By reflection, we may extend v to a harmonic function in B_2 and thus estimate (1.68) follows from the interior derivative estimate for harmonic functions (cf. Theorem 2.10 in [13]). □

Theorem 1.36. *Let B be a ball in $\mathbb{R}^n$ and u and f functions on $\bar{B}$ satisfying $u \in C^2(B) \cap C(\bar{B})$, $f \in C^\alpha(\bar{B})$ and*

$$\begin{cases} -\Delta u = f & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

then $u \in C^{2,\alpha}(\bar{B})$.

Proof. By translation invariance, we may assume ∂B passes through the origin. The inversion mapping $x \mapsto x^* := x/|x|^2$ is a bicontinuous and smooth mapping of the punctured space $\mathbb{R}^n \backslash \{0\}$ onto itself which maps B onto a half-space, B^*. Moreover, since $u \in C^2(B) \cap C(\bar{B})$, the Kelvin transform of u, i.e.,

$$v(x) = \frac{1}{|x|^{n-2}} u\Big(\frac{x}{|x|^2}\Big),$$

belongs to $C^2(B^*) \cap C(\bar{B}^*)$ and satisfies

$$-\Delta v(x^*) = |x^*|^{-(n+2)} f(\frac{x^*}{|x^*|^2}), \ x \in B^*.$$

Hence, we can apply Theorem 1.35 to the Kelvin transform v and since by translation invariance any point of ∂B may be re-centered to be the origin, we conclude that $u \in C^{2,\alpha}(\bar{B})$. □

We conclude now with an application of the boundary estimates to obtain an existence result for the Dirichlet problem.

Corollary 1.9. *Let $\varphi \in C^{2,\alpha}(\bar{B})$, $f \in C^\alpha(\bar{B})$. Then the Dirichlet problem*

$$\begin{cases} -\Delta u = f & \text{in } B, \\ u = \varphi & \text{on } \partial B, \end{cases}$$

is uniquely solvable for a function $u \in C^{2,\alpha}(\bar{B})$.

Proof. Writing $v = u - \varphi$, the problem is reduced to solving the problem

$$\begin{cases} -\Delta v = f - \Delta\varphi & \text{in } B, \\ v = 0 & \text{on } \partial B, \end{cases}$$

which is solvable for $v \in C^2(B) \cap C(\bar{B})$ by the usual representation formula via Green's functions and consequently for $v \in C^{2,\alpha}(\bar{B})$ by Theorem 1.36. □

1.4.5 A Glimpse at Hölder Regularity for General Equations

The Hölder regularity estimates above can be extended from the Laplace operator to more general elliptic operators. We state the extension of the results here, however, we shall revisit them and offer their proofs in Chapter 3.

As we have seen already, one of the primary motivations for establishing Hölder a priori estimates, or the so-called Schauder regularlity estimates, lies in its importance in generating existence results for boundary value problems. So, we state the a priori estimates early here as we will soon need them in the next chapter, which develops various existence results for many different elliptic problems.

We start by considering the elliptic equation

$$Lu = f \ \text{ in } U, \tag{1.70}$$

where as usual L is in divergence form, i.e.,

$$Lu = -\sum_{i,j=1}^{n} D_j \left(a^{ij}(x) D_i u\right) + \sum_{j=1}^{n} b^i(x) D_i u + c(x)u.$$

We suppose $a^{ij}, b^i, c \in C^\alpha(\bar{U})$ for some $0 < \alpha < 1$, and take L to be uniformly elliptic, i.e., there exist $0 < \lambda \leq \Lambda$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } x \in U,\ \xi \in \mathbb{R}^n.$$

We also assume

$$\frac{1}{\lambda}\Big\{\sum_{i,j=1}^{n} |a^{ij}|_{\alpha;U} + \sum_{i=1}^{n} |b^i|_{\alpha;U} + |c|_{\alpha;U}\Big\} \leq \Lambda_\alpha.$$

Then the following interior a priori estimate holds.

Theorem 1.37 (Interior Schauder estimates 0)**.** *For $\alpha \in (0,1)$, let $u \in C^{2,\alpha}(U)$ be a solution of* (1.70)*. Then for $U' \subset\subset U$, we have*

$$\|u\|_{2,\alpha;U'} \leq C\left(\frac{1}{\lambda}\|f\|_{\alpha;U} + \|u\|_{0;U}\right),$$

where C depends only on $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ and $dist(U', \partial U)$.

If the boundary of our domain U is nice enough, we can extend the $C^{2,\alpha}$ estimates up to the boundary.

Theorem 1.38 (Global Schauder estimates)**.** *Consider the same assumptions from the previous theorem and further assume $\partial U \in C^{2,\alpha}$. Suppose that $u \in C^{2,\alpha}(\bar{U})$ is a solution of* (1.70) *satisfying the boundary condition $u = g$ on ∂U where $g \in C^{2,\alpha}(\bar{U})$. Then*

$$\|u\|_{2,\alpha;U} \leq C\left(\frac{1}{\lambda}\|f\|_{\alpha;U} + \|g\|_{2,\alpha;U} + \|u\|_{0;U}\right),$$

where C depends only on $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ and U. Moreover, if u satisfies the maximum principle, then the last term on the right-hand side of the global estimate can be dropped.

CHAPTER 2

Existence Theory

This chapter reviews various methods and techniques for solving both linear and nonlinear elliptic problems. We will focus mainly on Dirichlet boundary value problems or elliptic problems defined on the entire space, e.g., $\mathbb{R}^n$.

2.1 The Lax-Milgram Theorem

Theorem 2.1 (Lax–Milgram)**.** *Let H be a Hilbert space with norm $\|\cdot\|$ and $B : H \times H \longrightarrow \mathbb{R}$ is a bilinear form. Suppose that there exist numbers α, $\beta > 0$ such that for any u, $v \in H$*

(i) Boundedness: $|B[u, v]| \le \alpha \|u\| \cdot \|v\|$,

(ii) Coercivity: $\beta \|u\|^2 \le B[u, u]$,

then for each $f \in L^2(U)$ there exists a unique $u \in H$ such that

$$B[u, v] = (f, v) \text{ for all } v \in H.$$

To prove the theorem, we first recall the Riesz representation theorem for Hilbert spaces.

Theorem 2.2 (Riesz representation)**.** *If f is a bounded linear functional on a Hilbert Space H with inner product $(\cdot, \cdot)$, then there exists an element $v \in H$ such that $< f, u >= (v, u)$ for all $u \in H$.*

It is clear that the inner product is a bilinear form which satisfies both the requirements of the Lax–Milgram theorem. However, the Lax–Milgram theorem is a stronger result than the Riesz representation theorem in that it does not require the bilinear form to be symmetric.

Proof. Existence: For each fixed $w \in H$, $v \longrightarrow B[w, v]$ is a bounded linear functional on H. By the Riesz representation theorem, there exists a $u \in H$ such that $(u, v) = B[w, v]$ for all $v \in H$. We define the operator $A : H \longrightarrow H$ by $u = A[w]$.
Step 1: Claim that $A : H \longrightarrow H$ is a bounded linear operator: To prove A is linear, observe that

$$\begin{aligned}(A[\lambda_1 u_1 + \lambda_2 u_2], v) &= B[\lambda_1 u_1 + \lambda_2 u_2, v] = \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &= (\lambda_1 A[u_1] + \lambda_2 A[u_2], v) \text{ for all } v \in H.\end{aligned}$$

Thus, $A[\lambda_1 u_1 + \lambda_2 u_2] = \lambda_1 A[u_1] + \lambda_2 A[u_2]$.

Moreover, A is bounded since

$$\|Au\|^2 = (Au, Au) = B[Au, u] \leq \alpha\|u\| \cdot \|Au\|$$

Hence, $\|Au\| \leq \alpha\|u\|$.

Step 2: Claim $Ran(A)$ is closed in H.
Let $\{y_k\}$ be a convergent sequence in $ran(A)$ so that there is a sequence $\{u_k\} \subset H$ for which $y_k = A[u_k] \longrightarrow y \in H$. By coercivity, $\|u_k - u_j\| \leq \beta\|A[u_k] - A[u_j]\|$, which implies $\{u_k\}$ is a Cauchy sequence in H. Hence, u_k converges to some element $u \in H$ and $y = A[u]$; that is, $y \in Ran(A)$, thereby proving $Ran(A)$ is closed in H.

Step 3: Claim $Ran(A) = H$.
On the contrary, assume that $Ran(A) \neq H$. Thus, we have that $H = ran(A) \oplus ran(A)^\perp$ since $Ran(A)$ is closed, and we choose a non-zero element $z \in ran(A)^\perp$. By the coercivity condition, $\beta\|z\|^2 \leq B[z, z] = (Az, z) = 0$ and we arrive at a contradiction.

Step 4: For each $f \in L^2$, the Riesz representation theorem once again implies there exists an element $z \in H$ for which $(z, v) =< f, v >$ for all $v \in H$. In turn, we can find a u such that $z = A[u]$, i.e., $(z, v) = (Au, v) = B[u, v]$ for all $v \in H$. Hence, we have found an element $u \in H$ for which $B[u, v] = (f, v)$ for all $v \in H$.

Uniqueness: Suppose that u_1 and u_2 are two such elements satisfying $B[u_1, v] = (f, v)$ and $B[u_2, v] = (f, v)$ for all $v \in H$, respectively. This implies that $B[u_1 - u_2, v] = 0$ for all $v \in H$. Now, if $v = u_1 - u_2$, the coercivity condition implies $\beta\|u_1 - u_2\|^2 \leq B[u_1 - u_2, u_1 - u_2] = 0$. Hence, $u_1 = u_2$. □

2.1.1 Existence of Weak Solutions

Our goal here is to prove existence and uniqueness of weak solutions to the Dirichlet boundary value problem of the following form:

$$\begin{cases} Lu + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2.1}$$

where μ is a non-negative constant to be determined later. Developing this result relies mainly on certain energy estimates and the Lax-Milgram theorem. In addition, we will now focus strictly on the second order differential operator in divergence form with its associated bilinear form

$$B[u,\, v] := \int_U \sum_{i,j=1}^{n} a^{ij}(x) D_i u D_j v + \sum_{i=1}^{n} b^i(x) D_i u v + c(x) uv \, dx,$$

and assume that $a^{ij}, b^i, c \in L^\infty(U)$ for $i, j = 1, ..., n$. Furthermore, assume U is an open and bounded subset of $\mathbb{R}^n$ and denote $H := H_0^1(U)$.

Energy Estimates

Theorem 2.3. *There exists constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that*

(i) $|B[u,\, v]| \leq \alpha \|u\|_H \|v\|_H$

(ii) $\beta \|u\|_H^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2$ *for all* $u, v \in H$.

Proof. We prove the first estimate of the theorem.

$$\begin{aligned} |B[u,\, v]| &= \Big| \int_U \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} v + cuv \, dx \Big| \\ &\leq \sum_{i,j=1}^{n} \|a^{ij}\|_{L^\infty} \int_U |Du \cdot Dv| \, dx + \sum_{i=1}^{n} \|b^i\|_{L^\infty} \int_U |Du||v| \, dx + \|c\|_{L^\infty} \int_U |u||v| \, dx, \end{aligned}$$

since it was assumed that $a^{ij}, b^i, c \in L^\infty(U)$. Now apply Hölder's inequality sufficiently many times and use the definition of the H-norm to get

$$|B[u,\, v]| \leq C \|u\|_H \|v\|_H$$

for some constant C.

To prove the second part, the definition of (uniform) ellipticity will be used. By uniform ellipticity, there is some $\lambda > 0$ such that

$$\begin{aligned} \lambda \int_U |Du|^2 \, dx &\leq \int_U \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i} u_{x_j} \, dx = B[u, u] - \int_U \sum_{i=1}^{n} b^i(x) u_{x_i} u + cu^2 \, dx \\ &\leq B[u, u] + \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)} \int_U |Du||u| \, dx + \|c\|_{L^\infty(U)} \int_U u^2 \, dx \end{aligned} \tag{2.2}$$

Using the Cauchy's inequality with ϵ i.e $ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, a, b > 0, \epsilon > 0$, we have

$$|Du||u| \leq \epsilon |Du|^2 + \frac{u^2}{4\epsilon} \Longrightarrow \int_U |Du||u| \, dx \leq \epsilon \int_U |Du|^2 \, dx + \frac{1}{4\epsilon} \int_U u^2 \, dx.$$

We may choose $\epsilon > 0$ such that $\epsilon \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)} < \frac{\lambda}{2}$, then plugging this back into (2.2) yields

$$\begin{aligned}\lambda \int_U |Du|^2\,dx &\leq B[u,u] + (\sum_{i=1}^{n} \|b^i\|_{L^\infty(U)})(\epsilon \int_U |Du|^2\,dx + \frac{1}{4\epsilon}\int_U u^2\,dx) + \|c\|_{L^\infty(U)} \int_U u^2\,dx \\ &\leq B[u,u] + \frac{\lambda}{2}\int_U |Du|^2\,dx + \Big((\sum_{i=1}^{n} \|b^i\|_{L^\infty(U)})\frac{1}{4\epsilon} + \|c\|_{L^\infty(U)}\Big)\int_U u^2\,dx.\end{aligned}$$

Now some rearrangement of terms yields

$$\frac{\lambda}{2}\int_U |Du|^2\,dx \leq B[u,u] + C\int_U u^2\,dx.$$

Adding $\frac{\lambda}{2}\int_U |u|^2\,dx$ on both sides of this inequality gives us our desired result,

$$\frac{\lambda}{2}\|u\|_H^2 \leq B[u,u] + \left(C + \frac{\lambda}{2}\right)\|u\|_{L^2(U)}^2.$$

□

Remark 2.1. *From our estimate (ii), we see that $B[\cdot,\cdot]$ does not directly satisfy the hypotheses of the Lax-Milgram theorem whenever $\gamma > 0$. Our next theorem will take this into consideration as it provides our existence and uniqueness result for the Dirichlet boundary value problem.*

Theorem 2.4 (First Existence Theorem for weak solutions)**.** *There is a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each function $f \in L^2(U)$, there exists a unique weak solution $u \in H = H_0^1(U)$ of the Dirichlet boundary value problem*

$$\begin{cases} Lu + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2.3}$$

Proof. Let γ be the same from the previous theorem, let $\mu \geq \gamma$ and define the bilinear form $B_\mu[u,v] = B[u,\ v] + \mu(u,v)_{L^2}$ with $u, v \in H$.
Claim: The bilinear form $B_\mu[\cdot,\cdot]$ satisfies the hypotheses of the Lax-Milgram theorem. More precisely, we have the bilinear estimate,

$$\begin{aligned}|B_\mu[u,v]| &= |B[u,\ v] + \mu(u,v)_{L^2}| \leq |B[u,\ v]| + \mu|(u,v)_{L^2}| \\ &\leq C\|u\|_H\|v\|_H + \mu\|u\|_{L^2}\|v\|_{L^2} \\ &\leq C\|u\|_H\|v\|_H,\end{aligned}$$

where in the second line we used the previous theorem and the Cauchy-Swharz inequality. Moreover, we have the coercivity estimate,

$$\begin{aligned}B_\mu[u,u] &= B[u,u] + \mu(u,u)_{L^2} \\ &\geq B[u,u] + \gamma(u,u)_{L^2} \\ &\geq C\|u\|_H,\end{aligned}$$

where we used the second bound from the energy estimates.

Now fix $f \in L^2(U)$ and set $\varphi_f(v) = (f, v)_{L^2}$. This is a bounded linear functional since, by the Cauchy-Schwarz inequality,

$$|\varphi_f(v)| = |(f, v)_{L^2}| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_H.$$

Thus, by the Lax-Milgram theorem, we can find a unique $u \in H$ satisfying $B_\mu[u, v] = \varphi_f(v)$ for all $v \in H$. That is, $u \in H$ is a unique weak solution to the Dirichlet boundary value problem. □

2.2 The Fredholm Alternative

First, we recall the Fredholm theory for compact operators then apply it to further develop our existence theory for second-order elliptic equations. Let X and Y be Banach spaces, H denotes a real Hilbert space with inner product $(\cdot, \cdot)$, and the operator L is the usual second order elliptic operator in divergence form.

Definition 2.1. *A bounded linear operator $K : X \longrightarrow Y$ is called* **compact** *provided each bounded sequence $\{u_k\}_{k=1}^{\infty} \subset X$, the sequence $\{Ku_k\}_{k=1}^{\infty}$ is precompact in Y, i.e., there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ such that $\{Ku_{k_j}\}_{j=1}^{\infty}$ converges in Y.*

Theorem 2.5 (Fredholm Alternative)**.** *Let $K : H \longrightarrow H$ be a compact linear operator. Then*

(a) The kernel $N(I - K)$ is finite dimensional,

(b) The range $R(I - K)$ is closed,

(c) $R(I - K) = N(I - K^*)^\perp$,

(d) $N(I - K) = \{0\}$ if and only if $R(I - K) = H$.

Remark 2.2. *This theorem basically asserts the following dichotomy, i.e., either*

(α) For each $f \in H$, the equation $u - Ku = f$ has a unique solution; or else

(β) the homogeneous equation $u - Ku = 0$ has non-trivial solutions.

Further, should (β) hold, the space of solutions of this homogeneous equation is finite dimensional, and the non-homogeneous equation

(γ) $u - Ku = f$ has a solution if and only if $f \in N(I - K^*)^\perp$.

We shall also require the following basic result on the spectrum of compact linear operators.

Theorem 2.6 (Spectrum of a compact operator)**.** *Assume* $\dim(H) = \infty$ *and* $K : H \longrightarrow H$ *is a compact linear operator. Then*

(i) $0 \in \sigma(K)$,

(ii) $\sigma(K)\backslash\{0\} = \sigma_p(K)\backslash\{0\}$,

(iii) $\sigma(K)\backslash\{0\}$ *is finite, or else is a sequence tending to* 0.

2.2.1 Existence of Weak Solutions

Definition 2.2. *We define the following.*

(a) The operator L^**, the formal adjoint of* L*, is*

$$L^*v := -\sum_{i,j=1}^{n}(a^{ij}(x)v_{x_j})_{x_i} - \sum_{i=1}^{n} b^i(x)v_{x_i} + \Big(c(x) - \sum_{i=1}^{n} b^i_{x_i}(x)\Big)v,$$

provided $b^i \in C^1(\bar{U})$, $i = 1, 2, \ldots, n$.

(b) The adjoint bilinear form $B^* : H_0^1(U) \times H_0^1(U) \longrightarrow \mathbb{R}$ *is defined by*

$$B^*[v, u] := B[u, v]$$

for all $u, v \in H_0^1(U)$.

(c) We say that $v \in H_0^1(U)$ *is a weak solution of the adjoint problem*

$$\begin{cases} L^*v = f & \text{in } U, \\ \quad v = 0 & \text{on } \partial U, \end{cases}$$

provided that

$$B^*[v, u] = (f, u)$$

for all $u \in H_0^1(U)$.

Theorem 2.7 (Second Existence Theorem for weak solutions)**.** *There holds the following.*

(a) Precisely one of the following statements holds:

(α) For each $f \in L^2(U)$ *there exists a unique weak solution* u *of the boundary value problem*

$$\begin{cases} Lu = f & \text{in } U, \\ \quad u = 0 & \text{on } \partial U, \end{cases} \tag{2.4}$$

or else

(β) there exists a weak solution $u \not\equiv 0$ of the homogeneous problem

$$\begin{cases} Lu = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2.5}$$

(b) Furthermore, should assertion (β) hold, the dimension of the subspace $N \subset H_0^1(U)$ of weak solutions of (2.5) *is finite and equals the dimension of the subspace $N^* \subset H_0^1(U)$ of weak solutions of*

$$\begin{cases} L^*v = 0 & \text{in } U, \\ v = 0 & \text{on } \partial U. \end{cases} \tag{2.6}$$

(c) Finally, the boundary value problem (2.4) *has a weak solution if and only if*

$$(f, v) = 0 \ \text{ for all } \ v \in N^*.$$

Proof. **Step 1:** As in the proof of Theorem 2.4, choose $\mu = \gamma$ and define the bilinear form

$$B_\gamma[u, v] := B[u, v] + \gamma(u, v),$$

corresponding to the operator $L_\gamma u := Lu + \gamma u$. Thus, for each $g \in L^2(U)$, there exists a unique $u \in H_0^1(U)$ solving

$$B_\gamma[u, v] = (g, v) \ \text{ for all } \ v \in H_0^1(U). \tag{2.7}$$

Write $u = L_\gamma^{-1} g$ whenever (2.7) holds.
Step 2: Observe that $u \in H_0^1(U)$ is a weak solution of (2.4) if and only if

$$B_\gamma[u, v] = (\gamma u + f, v) \ \text{ for all } \ v \in H_0^1(U), \tag{2.8}$$

that is, if and only if

$$u = L_\gamma^{-1}(\gamma u + f). \tag{2.9}$$

We can rewrite this as

$$u - Ku = h, \tag{2.10}$$

where $Ku := \gamma L_\gamma^{-1} u$ and $h := L_\gamma^{-1} f$.
Step 3: We now claim that $K : L^2(U) \longrightarrow L^2(U)$ is a bounded, linear, compact operator. Indeed, from our choice of γ and the energy estimates from the previous section, we note that if (2.7) holds, then

$$\beta \|u\|^2_{H_0^1(U)} \le B_\gamma[u, u] = (g, u) \le \|g\|_{L^2(U)} \|u\|_{L^2(U)} \le \|g\|_{L^2(U)} \|u\|_{H_0^1(U)},$$

and so

$$\|Kg\|_{L^2(U)} \leq \|Kg\|_{H_0^1(U)} = \|\gamma L_\gamma^{-1} g\|_{H_0^1(U)} = \|u\|_{H_0^1(U)} \leq C\|g\|_{L^2(U)} \text{ for } g \in L^2(U)$$

for some suitable constant $C > 0$. However, since $H_0^1(U) \subset\subset L^2(U)$ by the Rellich-Kondrachov compactness theorem (see Theorem A.22), we conclude that K is a compact operator.

Step 4: By the Fredholm alternative, we conclude either

(α) for each $h \in L^2(U)$ the equation $u - Ku = h$ has a unique solution $u \in L^2(U)$; or else

(β) the equation $u - Ku = 0$ has non-trivial solutions in $L^2(U)$.

Should assertion (α) hold, then according to (2.8)–(2.10), there exists a unique weak solution of problem (2.4). On the other hand, should assertion (β) be valid, then necessarily $\gamma \neq 0$ and we recall that the dimension of the space N of the solutions of $u - Ku = 0$ is finite and equals the dimension of the space N^* of solutions of the equation

$$v - K^* v = 0. \tag{2.11}$$

However, we have that (β) holds if and only if u is a weak solution of (2.5) and that (2.11) holds if and only if v is a weak solution of (2.6).

Step 5: Finally, we recall equation $u - Ku = h$ in (α) has a solution if and only if

$$(h, v) = 0$$

for all v solving (2.11). However, from (2.11) we compute that

$$(h, v) = \frac{1}{\gamma}(Kf, v) = \frac{1}{\gamma}(f, K^* v) = \frac{1}{\gamma}(f, v).$$

Hence, the boundary value problem (2.4) has a solution if and only if $(f, v) = 0$ for all weak solutions v of (2.6).

□

Definition 2.3. *We say $\lambda \in \Sigma$, the (real) spectrum of the operator L, if the boundary value problem*

$$\begin{cases} Lu = \lambda u & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

has a non-trivial solution w, in which case λ is called an **eigenvalue** *of L, w a corresponding* **eigenfunction**. *Particularly, the partial differential equation $Lu = \lambda u$ for $L = -\Delta$ is often called the Helmholtz equation.*

Theorem 2.8 (Third Existence Theorem for weak solutions)**.** *There holds the following.*

(a) *There exists an at most coutable set* $\Sigma \subset \mathbb{R}$ *such that the boundary value problem*

$$\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2.12}$$

has a unique weak solution for each $f \in L^2(U)$ *if and only if* $\lambda \notin \Sigma$.

(b) *If* Σ *is infinite, then* $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$*, the values of a non-decreasing sequence with* $\lambda_k \longrightarrow \infty$.

Proof. **Step 1:** Let γ be the constant from Theorem 2.3 and assume $\lambda > -\gamma$. Without loss of generality, we also assume $\gamma > 0$.

According to the Fredholm alternative, problem (2.12) has a unique weak solution for each $f \in L^2(U)$ if and only if $u \equiv 0$ is the only weak solution of the homogeneous problem

$$\begin{cases} Lu = \lambda u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

This is in turn true if and only if $u \equiv 0$ is the only weak solution of

$$\begin{cases} Lu + \gamma u = (\gamma + \lambda) u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2.13}$$

Now (2.13) holds precisely when

$$u = L_\gamma^{-1}(\gamma + \lambda) u = \frac{\gamma + \lambda}{\gamma} K u, \tag{2.14}$$

where, as in the proof of the previous theorem, $Ku := \gamma L_\gamma^{-1} u$ and K is a bounded and compact linear operator on $L^2(U)$.

Now, if $u \equiv 0$ is the only solution of (2.14), we see

$$\frac{\gamma}{\gamma + \lambda} \text{ is not an eigenvalue of } K. \tag{2.15}$$

Hence, we see that (2.12) has a unique weak solution for each $f \in L^2(U)$ if and only if (2.15) holds.

Step 2: According to Theorem 2.6, the set of all non-zero eigenvalues of K forms either finite set or else the values of a sequence converging to zero. In the second case, $\lambda > -\gamma$ and (2.14) imply that (2.12) has a unique weak solution for all $f \in L^2(U)$ except for a sequence $\lambda_k \longrightarrow \infty$. □

Theorem 2.9 (Boundedness of the inverse)**.** *If* $\lambda \notin \Sigma$*, there exists a positive constant* C *such that*

$$\|u\|_{L^2(U)} \leq C \|f\|_{L^2(U)},$$

whenever $f \in L^2(U)$ *and* $u \in H_0^1(U)$ *is the unique weak solution of*

$$\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

The constant C *depends only on* λ, U*, and the coefficients of the elliptic operator* L.

2.3 Eigenvalues and Eigenfunctions

This section is somewhat of a digression from the rest of the chapter in that we study eigenvalues for symmetric uniformly elliptic operators. We feel that this follows naturally from the previous section as we continue to examine properties of compact operators in the setting of partial differential equations. As such, we only consider symmetric elliptic operators, but the theory certainly extends to the non-symmetric setting (see [9]).

We consider the boundary value problem

$$\begin{cases} Lw = \lambda w & \text{in } U, \\ w = 0 & \text{on } \partial U, \end{cases} \tag{2.16}$$

where $U \subset \mathbb{R}^n$ is open, bounded and connected. We say $\lambda \in \mathbb{C}$ is an eigenvalue of L provided there exists a non-trivial solution w of problem (2.16) where w is called the corresponding eigenfunction of λ. As we shall see, L is a compact and symmetric linear operator (actually it is really the inverse operator L^{-1} that satisfies these properties) and therefore, elementary spectral theory indicates the spectrum Σ of L is positive, real and at most countable. In particular, we take L to be of the form

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j},$$

where $a^{ij} \in C^\infty(\bar{U})$ and $a^{ij} = a^{ji}$ for $i, j = 1, 2, \ldots, n$. We note that the associated bilinear form $B[\cdot, \cdot]$ associated with this eigenvalue problem is symmetric, i.e., $B[u, v] = B[v, u]$ for all $u, v \in H_0^1(U)$ since L is formally symmetric.

Theorem 2.10 (Eigenvalues of symmetric elliptic operators)**.** *There hold the following.*

(a) Each eigenvalue of L is real.

(b) Furthermore, if we repeat each eigenvalue according to its finite multiplicity, we have

$$\Sigma = \{\lambda_k\}_{k=1}^{\infty}$$

where

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots.$$

and $\lambda_k \longrightarrow \infty$ as $k \longrightarrow \infty$.

(c) Finally, there exists an orthonormal basis $\{w_k\}_{k=1}^{n}$ of $L^2(U)$ where $w_k \in H_0^1(U)$ is an eigenvalue corresponding to λ_k in (2.16).

Remark 2.3. *The first eigenvalue $\lambda_1 > 0$ is often called the* **principal eigenvalue** *of L. Moreover, as examined in the next chapter, basic regularity theory ensures the eigenfunctions w_k, for $k = 1, 2, \ldots,$ actually belong to $C^\infty(U)$. In fact, they belong to $C^\infty(\bar{U})$ provided that the boundary ∂U is smooth.*

Proof. In fact, it is simple to show that $S = L^{-1}$ is a bounded and compact linear operator on $L^2(U)$. More precisely, for $f \in L^2(U)$, $Sf = u$ means $u \in H_0^1(U)$ is the weak solution of

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

Now, we claim that S is also symmetric. To see this, let $f, g \in L^2(U)$ and take $Sf = u$ and $Sg = v$. Notice that

$$(Sf, g) = (u, g) = B[u, v]$$

and

$$(f, Sg) = (f, v) = B[u, v].$$

Hence, the basic theory of compact, symmetric linear operators on Hilbert spaces imply the eigenvalues of S are real, positive and its corresponding eigenfunctions make up an orthonormal basis of $L^2(U)$. Moreover, for $\eta \neq 0$ and $\lambda = \eta^{-1}$, there holds $Sw = \eta w$ if and only if $Lw = \lambda w$. Thus, the same properties translate to the eigenvalues and eigenfunctions of L as well. This completes the proof.

□

Theorem 2.11 (Variational principle for the principal eigenvalue)**.** *There hold the following statements.*

(a) Rayleigh's formula holds, i.e.,

$$\lambda_1 = \min_{\|u\|_{L^2(U)}} \{B[u, u] \mid u \in H_0^1(U)\} = \min_{u \neq 0 \text{ in } H_0^1(U)} \frac{B[u, u]}{\|u\|_{L^2(U)}^2}.$$

(b) Furthermore, the above minimum is attained by a function $w_1 \in H_0^1(U)$, positive within U, which is also a weak solution of

$$\begin{cases} Lu = \lambda_1 u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

(c) The principle eigenvalue is simple, i.e., if $u \in H_0^1(U)$ is any weak solution of

$$\begin{cases} Lu = \lambda_1 u & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

then u is a multiple of w_1. Therefore, the eigenvalues of L can be ordered as follows:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots.$$

2.4 Topological Fixed Point Theorems

This section introduces topological fixed point theorems from functional analysis to establish the existence of weak solutions to a class of nonlinear elliptic PDEs.

2.4.1 Brouwer's Fixed Point Theorem

Before stating and proving Schauder's fixed point theorem, we state and prove Brouwer's fixed point theorem, since we will need it to prove Schauder's version. In particular, Schauder's theorem will be a generalization of Brouwer's to infinite dimensional Banach spaces. We adopt the notation that $B_r(x)$ or $B(x,r) \subset \mathbb{R}^n$ to represent the ball of radius r with center $x \in \mathbb{R}^n$, and we denote its closure by $\bar{B}_r(x)$ or $\bar{B}(x,r)$, respectively.

Theorem 2.12 (Brouwer's Fixed Point Theorem)**.** *Assume $u : \bar{B}_1(0) \to \bar{B}_1(0)$ is continuous. Then u has a fixed point, that is, there exists a point $x \in \bar{B}_1(0))$ with $u(x) = x$.*

To prove this, we exploit the fact that the unit sphere is not a retract of the closed unit ball. Namely, we prove

Theorem 2.13 (No Retraction Theorem)**.** *There is no continuous function*

$$u : \bar{B}_1(0) \longrightarrow \partial B_1(0)$$

such that $u \equiv Identity$ on $\partial B_1(0)$.

Proof. We proceed with a topological degree argument (see Chapter 1 in [24]). Assume that the unit sphere is a retract of the closed unit ball and a retraction mapping is given by u. Then, homotopy invariance ensures that $deg(u, B_1(0), 0) = deg(Identity, B_1(0), 0) = 1$ and thus there exists an interior point $x \in B_1(0)$ such that $u(x) = 0$. This is a contradiction with the assumption that $u(\bar{B}_1(0)) \subseteq \partial B_1(0)$. □

Proof of Brouwer's Fixed Point Theorem. Assume that $u(x) \neq x$ for all $x \in \bar{B}_1(0)$. Thus, we can define a map $w : \bar{B}_1(0) \longrightarrow \partial B_1(0)$ by letting w be the intersection of $\partial B_1(0)$ with the straight line starting at $u(x)$ and passing through x and ending on the boundary. This terminal boundary point is equal to $w(x)$, or more precisely,

$$w(x) = x + \gamma(u(x) - x),$$

where $\gamma = \gamma(x)$ is a real-valued map that ensures that $w(x)$ has unit norm. Clearly, w is continuous and $w(x) = x$ for all $x \in \partial B_1(0)$. Therefore, this implies that the unit sphere is a retract of the closed unit ball and we arrive at a contradiction with Theorem 2.13. This completes the proof of the theorem. □

Remark 2.4. *Brouwer's fixed point theorem generalizes to compact and convex subsets, since such proper subsets with non-empty interior are homeomorphic to the closed unit ball.*

2.4.2 Schauder's Fixed Point Theorem

Let us consider a Banach space X with norm $\|\cdot\|$.

Theorem 2.14 (Schauder)**.** *Suppose that K is a compact and convex subset of X. Assume that $A : K \to K$ is continuous. Then A has a fixed point in K.*

Proof. **Step 1:** Fix $\epsilon > 0$. Since K is compact, we can choose finitely many points $u_1, u_2, ..., u_{N_\epsilon}$ so that the collection of open balls $\{B(u_i, \epsilon)\}_{i=1}^{N_\epsilon}$ is a cover for K, i.e., $K \subset \bigcup_{i=1}^{N_\epsilon} B(u_i, \epsilon)$. Now let K_ϵ be the closed convex hull of the points $\{u_1, u_2, ..., u_{N_\epsilon}\}$:

$$K_\epsilon = \Big\{ \textstyle\sum_{i=1}^{N_\epsilon} \lambda_i u_i \,|\, 0 \le \lambda_i \le 1, \sum_{i=1}^{N_\epsilon} \lambda_i = 1 \Big\}.$$

So $K_\epsilon \subset K$ from the convexity of K and by definition of K_ϵ.

Let us define the operator $P_\epsilon : K \longrightarrow K$ by

$$P_\epsilon(u) = \frac{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))u_i}{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))} \text{ for } u \in K.$$

Remark 2.5. *We define the distance of $x \in X$ from a subset $Y \subset X$ by*

$$dist(x, Y) = \inf_{y \in Y} dist(x, y) = \inf_{y \in Y} \|x - y\|.$$

$P_\epsilon : K \longrightarrow K$ is well-defined since the denominator $\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))$ is never zero since $K \subset \bigcup_{i=1}^{N_\epsilon} B(u_i, \epsilon)$, i.e., u belongs to at least one of the open balls in the cover.

Step 2: In addition, $P_\epsilon : K \longrightarrow K$ is continuous. Suppose $\{v_k\} \longrightarrow v$ in K. Define for each $j = 1, ..., N_\epsilon$ the operator $P_\epsilon^j : K \longrightarrow K$ by

$$P_\epsilon^j(u) = \frac{dist(u, K - B(u_j, \epsilon))u_j}{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))} \text{ for } u \in K.$$

Then for some constant M,

$$\begin{aligned} \|P_\epsilon^j(v_k) - P_\epsilon^j(v)\| &\le M \cdot \inf_{y \in K - B(u_j)} |\|v_k - y\| - \|v - y\|| \\ &\le M \cdot \inf_{y \in K - B(u_j)} \|v_k - v\| \longrightarrow 0 \text{ as } k \longrightarrow \infty. \end{aligned}$$

Hence, each P_ϵ^j is continuous so therefore

$$P_\epsilon = \sum_{j=1}^{N_\epsilon} P_\epsilon^j$$

is continuous. Moreover, for $u \in K$ we have

$$\begin{aligned} \|P_\epsilon(u) - u\| &= \Big\| \frac{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))u_i}{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))} - u \Big\| \le \Big\| \frac{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))(u_i - u)}{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))} \Big\| \\ &\le \frac{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))\|u_i - u\|}{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))} \le \|u_i - u\| \le \epsilon. \end{aligned}$$

Step 3: Now consider the operator $A_\epsilon : K_\epsilon \longrightarrow K_\epsilon$ defined by $A_\epsilon[u] := P_\epsilon[A(u)]$, $(u \in K_\epsilon)$. As remarked earlier, we note that K_ϵ is homeomorphic to the closed unit ball $\bar{B}(0,1)$ in the Euclidean space $\mathbb{R}^{M_\epsilon}$ for some $M_\epsilon \leq N_\epsilon$. With this result, we can apply Brouwer's fixed point theorem to obtain the existence of a fixed point $u_\epsilon \in K_\epsilon$ with $A_\epsilon[u_\epsilon] = u_\epsilon$.

Step 4: We have that $\{u_\epsilon\}_{\epsilon>0}$ forms a sequence in K. The compactness of K implies that there is a subsequence, $\{u_{\epsilon_j}\}_{\epsilon_j>0}$, of $\{u_\epsilon\}_{\epsilon>0}$ that converges to some element $v \in K$. We now will show that this element v is in fact a fixed point of A. Using the bound from Step 2, one can establish that

$$\|u_{\epsilon_j} - A[u_{\epsilon_j}]\| = \|A_{\epsilon_j}[u_{\epsilon_j}] - A[u_{\epsilon_j}]\| = \|P_{\epsilon_j}[A[u_{\epsilon_j}]] - A[u_{\epsilon_j}]\| \leq \epsilon_j.$$

By utilizing the continuity of A, as $\epsilon_j \longrightarrow 0$ then the bound gives us that $\|v - Av\| \leq 0$ and thus $Av - v = 0$. □

2.4.3 Schaefer's Fixed Point Theorem

We shall deduce Schaefer's fixed point theorem from Schauder's. We shall see that this theorem is much more useful in application to PDEs since we work with compact operators rather than compact subsets of our Banach space X. However, before proceeding, we give two equivalent definitions on the notion of a compact operator or map.

Definition 2.4. *A (nonlinear) mapping $A : X \longrightarrow X$ on a Banach space X is compact if*

1. *for each bounded sequence $\{u_k\}_{k=1}^{\infty}$ in X, the sequence $\{A[u_k]\}_{k=1}^{\infty}$ is precompact, i.e., has a convergent subsequence in X.*

2. *for each bounded set $B \subset X$, $A(B)$ is precompact in X, i.e., its closure in X is a compact subset of X.*

Remark 2.6. *The former definition of sequential compactness was already provided in the previous section concerning the Fredholm alternative.*

Theorem 2.15 (Schaefer)**.** *Suppose $A : X \longrightarrow X$ is a continuous and compact mapping. Assume further that the set $S = \{u \in X \mid u = \lambda A[u]$, for some $0 \leq \lambda \leq 1\}$ is bounded. Then A has a fixed point in X.*

Proof. Suppose $u = \lambda A[u]$ for some $\lambda \in [0,1]$. Since S is bounded, we can find $M > 0$ such that $\|u\| < M$. Define $\bar{A} : \bar{B}(0,M) \longrightarrow \bar{B}(0,M)$ by

$$\bar{A}[u] = \begin{cases} A[u] & \text{if } \|A[u]\| \leq M, \\ \dfrac{M}{\|A[u]\|}A[u] & \text{if } \|A[u]\| \geq M. \end{cases} \tag{2.17}$$

Set K to be the closed convex hull of $\bar{A}(B(0,1))$. Since A is compact, and any scalar multiple of a compact operator is compact implies that $\bar{A}$ is compact as well. Using the result that the convex hull of a precompact set is precompact, we deduce that K is a convex, closed and precompact subset of X. Hence K is a compact and convex subset of X and $\bar{A} : K \longrightarrow K$ is a compact and continuous map. By Schauder's fixed point theorem, there exists a fixed point $u^* \in K$ with $\bar{A}[u^*] = u^*$.

We will now show that u^* is also a fixed point of A. Assume otherwise; so that $\|A[u^*]\| > 0$ and $u^* = \lambda A[u^*]$ with $\lambda = \frac{M}{\|A[u^*]\|} < 1$. However, $\|u^*\| = \|\bar{A}[u^*]\| = M$ since $\|\lambda A[u^*]\| = \frac{M\|A[u^*]\|}{\|A[u^*]\|} = \bar{A}[u^*] = M$, a contradiction.

□

2.4.4 Application to Nonlinear Elliptic Boundary Value Problems

We focus on solving a class of non-linear elliptic PDEs which can be treated as compact operators on some suitable function space. In such cases, Schaefer's fixed point theorem can be applied. We provide a fundamental example.

Consider the semilinear boundary-value problem

$$\begin{cases} -\Delta u + b(Du) + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2.18}$$

where U is a bounded and open subset of $\mathbb{R}^n$ and ∂U is smooth, $b : \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth and Lipschitz continuous so that

$$|b(p)| \leq C(|p| + 1)$$

for some positive constant C. We will prove the following claim.

Theorem 2.16. *If $\mu > 0$ is sufficiently large, there exists a function $u \in H_0^1(U)$ solving the boundary-value problem (2.18). Furthermore, u also belongs to $H^2(U)$.*

Proof. We prove the theorem in three main steps.

Step 1: Given $u \in H_0^1(U)$, set $f := -b(Du)$. So by Lipschitz continuity we can show $f \in L^2(U)$ since

$$|f(u)| = |b(Du)| \leq C(|Du| + 1),$$

then

$$\|f\|_{L^2(U)} \leq \|Du\|_{L^2(U)} + C \leq \|u\|_{H_0^1(U)} + C < \infty.$$

Now we will define the map $A : H_0^1(U) \to H_0^1(U)$. Formulate the linear boundary value problem

$$\begin{cases} -\Delta w + \mu w = f(u) & \text{in } U \\ w = 0 & \text{on } \partial U. \end{cases} \tag{2.19}$$

Since f was shown to belong to $L^2(U)$, linear PDE theory ensures the existence of a unique weak solution $w \in H_0^1(U)$ of the linear problem (2.19). Hence, for $u \in H_0^1(U)$, define $A[u] = w$. Moreover, basic elliptic regularity theory yields the estimate

$$\|w\|_{H^2(U)} = \|A[u]\|_{H^2(U)} \leq C\|f\|_{L^2(U)}$$

for some constant C (see Theorem 3.15 in the next chapter). Combining this with the above L^2 estimate on f, we get

$$\|w\|_{H^2(U)} = \|A[u]\|_{H^2(U)} \leq C(\|u\|_{H_0^1(U)} + 1)$$

for some constant C.

Step 2: We will show that $A : H_0^1(U) \to H_0^1(U)$ is a continuous and compact mapping. Suppose that $\{u_k\}_{k=1}^{\infty} \longrightarrow u$ in $H_0^1(U)$. Since

$$\|w\|_{H^2(U)} \leq C(\|u\|_{H_0^1(U)} + 1) \text{ for each } k \in \mathbb{N},$$

this implies that

$$\sup_k \|w_k\|_H^2(U) < \infty.$$

Then, as a consequence of the Rellich-Kondrachov compactness theorem (see Theorem A.22), there is a subsequence $\{w_{k_j}\}_{j=1}^{\infty}$ and a function $w \in H_0^1(U)$ with $\{w_{k_j}\}_{j=1}^{\infty} \longrightarrow w$ in $H_0^1(U)$. Note that each element of the subsequence satisfies $-\Delta w_{k_j} + \mu w_{k_j} = b(Du_{k_j})$. Now if we multiply this by any $v \in H_0^1(U)$ and integrate over U we obtain

$$\int_U -\Delta w_{k_j} v + \mu w_{k_j} v \, dx = -\int_U b(Du_{k_j}) v \, dx.$$

Integration by parts on the first term yields

$$\int_U Dw_{k_j} \cdot Dv + \mu w_{k_j} v \, dx = -\int_U b(Du_{k_j}) v \, dx.$$

Taking the limit as $j \longrightarrow \infty$ gives us

$$\int_U Dw \cdot Dv + \mu w v \, dx = -\int_U b(Du) v \, dx \text{ for all } v \in H_0^1(U).$$

This shows that $A[u] = w$ and $A[u_k] \longrightarrow A[u]$ in $H_0^1(U)$ given $u_k \longrightarrow u$ in $H_0^1(U)$. So A is a continuous map.

It is similar to show that A is compact. Take $\{u_k\}_{k=1}^{\infty}$ to be a bounded sequence in $H_0^1(U)$ We have already shown that $\sup_k \|w_k\|_{H^2(U)} < \infty$ so $\{A[u_k]\}_{k=1}^{\infty}$ is a bounded sequence in $H^2(U) \cap H_0^1(U)$; therefore it must contain a strongly convergent subsequence in $H_0^1(U)$. Again, this is a consequence of the Rellich-Kondrachov compactness theorem, which says that $H^2(U)$ is compactly embedded into $H_0^1(U)$.

Step 3: The final part to show is that if μ is sufficiently large, the set

$$S = \Big\{ u \in H_0^1(U) \,|\, u = \lambda A[u] \text{ for some } 0 < \lambda \leq 1 \Big\}$$

is a bounded set in $H_0^1(U)$. So let us assume $u \in S$ so that $u/\lambda = A[u]$ or $u \in H^2(U) \cap H_0^1(U)$ and $-\Delta u + \mu u = \lambda b(Du)$ a.e. in U. Multiply (2.18) by u then integrate over U to get

$$\begin{aligned}
\int_U (-\Delta + \mu u) u \, dx = \int_U Du \cdot Du + \mu |u|^2 \, dx &= \int_U |Du|^2 + \mu |u|^2 \, dx \\
&= -\int_U \lambda b(Du) u \, dx \leq \int_U |b(Du)||u| \, dx \leq \int_U C(|Du| + 1)|u| \, dx \\
&\leq \frac{1}{2} \int_U (|Du| + 1)^2 + C|u|^2 \, dx \leq \frac{1}{2} \int_U |Du|^2 + K \int_U |u|^2 + 1 \, dx
\end{aligned}$$

for some constants C and K independent of λ. This implies that

$$\frac{1}{2} \int_U |Du|^2 \, dx + (\mu - K) \int_U |u|^2 \, dx \leq K \int_U dx =: \frac{1}{2} M^2$$

where M is a positive constant. From our bounds, note that M is independent of the choice of $u \in S$. So if we choose

$$\mu = K + \frac{1}{2}$$

then

$$\frac{1}{2} \int_U |u|^2 + |Du|^2 \, dx \leq \frac{1}{2} M^2.$$

Hence, $\|u\|_{H_0^1(U)} \leq M < \infty$ for all $u \in S$, i.e., S is bounded in $H_0^1(U)$.

Finally apply Schaefer's fixed point theorem on $X = H_0^1(U)$ to show that A has a fixed point in $H^2(U) \cap H_0^1(U)$. By our construction of the mapping A, this fixed point solves our semilinear elliptic problem.

□

2.5 Perron Method

In this section, we introduce the Perron method to obtain the existence of classical solutions to Dirichlet problems on general domains provided that the solutions of the same problems on ball domains are known to exist. For simplicity and as our main example, we consider Laplace's equation on general domains. That is, let U be a bounded domain in $\mathbb{R}^n$ and φ be a continuous function on ∂U. Consider

$$\begin{cases} -\Delta u = 0 & \text{in } U, \\ \quad\;\; u = \varphi & \text{on } \partial U. \end{cases} \tag{2.20}$$

Note that, if U is an open ball, then the solutions of (2.20) are given by Poisson's formula via the Green's function on a ball domain. Otherwise, we shall use the Perron method in which the maximum principle plays an important role. First, we define continuous subharmonic and superharmonic functions based on the maximum principle.

Definition 2.5. *Let U be a bounded domain in $\mathbb{R}^n$ and v be a continuous function in U. Then v is subharmonic (respectively superharmonic) in U if for any ball $B \subset U$ and any harmonic function $w \in C(\bar{B})$,*

$$v \leq (\textit{respectively} \geq)\, w \;\text{ on }\; \partial B \;\text{ implies }\; v \leq (\textit{respectively} \geq)\, w \;\text{ in }\; B.$$

Before introducing the Perron method, we start with some preliminary results.

Lemma 2.1. *Let U be a bounded domain in $\mathbb{R}^n$ and $u, v \in C(\bar{U})$. Suppose u is subharmonic in U and v is superharmonic in U with $u \leq v$ on ∂U. Then $u \leq v$ in U.*

Proof. Without loss of generality, let us assume U is connected. Indeed, $u - v \leq 0$ on ∂U. Set $M = \max_{\bar{U}}(u - v)$ and

$$D = \{x \in U \,|\, u(x) - v(x) = M\} \subset U.$$

We claim that D is both an open and relatively closed subset of U and so, by the connectedness of U, either $D = \emptyset$ or $D = U$. It is clear that D is a relatively closed subset by the continuity of u and v. To show D is open, take any point $x_0 \in D$ and take $r < dist(x_0, \partial U)$. Let $\bar{u}$ and $\bar{v}$ solve, respectively,

$$\Delta\bar{u} = 0, \;\text{ in }\; B_r(x_0), \quad \bar{u} = u \;\text{ on }\; \partial B_r(x_0),$$
$$\Delta\bar{v} = 0, \;\text{ in }\; B_r(x_0), \quad \bar{v} = v \;\text{ on }\; \partial B_r(x_0).$$

Now, the existence of the solutions $\bar{u}$ and $\bar{v}$ is guaranteed by Poisson's formula for $U = B_r(x_0)$. Moreover, by recalling the definitions of subsolutions and supersolutions, we deduce that $u \leq \bar{u}$ and $\bar{v} \leq v$ in $B_r(x_0)$. Therefore,

$$\bar{u} - \bar{v} \geq u - v \;\text{ in }\; B_r(x_0).$$

Next,

$$\begin{cases} \Delta(\bar{u} - \bar{v}) = 0 & \text{in } B_r(x_0), \\ \quad \bar{u} - \bar{v} = u - v & \text{on } \partial B_r(x_0). \end{cases}$$

With $u - v \leq M$ on $\partial B_r(x_0)$, the maximum principle implies $\bar{u} - \bar{v} \leq M$ in $B_r(x_0)$. In particular,

$$M \geq (\bar{u} - \bar{v})(x_0) \geq (u - v)(x_0) = M.$$

Hence, $(\bar{u} - \bar{v})(x_0) = M$ and then $\bar{u} - \bar{v}$ has an interior maximum at x_0. Then, by the strong maximum principle, $\bar{u} - \bar{v} \equiv M$ in $B_r(\bar{x}_0)$, i.e., $u - v = M$ on $\partial B_r(x_0)$, and this holds for all $r < dist(x_0, \partial U)$. Then $u - v = M$ in $B_r(x_0)$ and thus $B_r(x_0) \subset D$. We conclude that $D = \emptyset$ or $D = U$, i.e., either $u - v$ attains its maximum only at ∂U or $u - v$ is constant in U. By $u \leq v$ in ∂U, we have $u \leq v$ in U in both cases. □

Remark 2.7. *In the proof above, we actually proved the strong maximum principle: Either $u < v$ in U or $u - v$ is constant in U.*

Lemma 2.2. *Let $v \in C(\bar{U})$ be a subharmonic function in U and $B \subset\subset U$ is a ball. Let w be defined by $w = v$ in $\bar{U} \backslash B$ and $\Delta w = 0$ in B. Then w is a subharmonic function in U and $v \leq w$ in $\bar{U}$.*

Remark 2.8. *Here, the function w is often called the* **harmonic lifting** *of v in B.*

Proof of Lemma 2.2. The existence of w is implied by Poisson's formula for $U = B$. Also, w is smooth in B and continuous in $\bar{U}$. We also have $v \leq w$ in B by definition of subharmonic functions in U. Now take any $B' \subset\subset U$ and consider a harmonic function $u \in C(\bar{B}')$ with $w \leq u$ on $\partial B'$. By $v \leq w$ on $\partial B'$, we have $v \leq u$ on $\partial B'$. Then, v is subharmonic and u is harmonic in B' with $v \leq u$ on $\partial B'$. By Lemma 2.1, we have $v \leq u$ in B'. Hence, $w \leq u$ in $B \backslash B'$. Additionally, both w and u are harmonic in $B \cap B'$ and $w \leq u$ on $\partial(B \cap B')$. So by the maximum principle, we have $w \leq u$ in $B \cap B'$. Hence, $w \leq u$ in B'. We then conclude that, by definition, w is subharmonic in U. This completes the proof of the lemma. □

Now we are ready to solve (2.20) via the Perron method. Set

$$u_\varphi(x) = \sup\{v(x) \,|\, v \in C(\bar{U}) \text{ is subharmonic in } U,\ v \leq \varphi \text{ on } \partial U\}. \tag{2.21}$$

Ultimately, our goal is to show that this function u_φ is indeed a solution of the Dirichlet problem (2.20). The first step in the Perron method is to show that u_φ in (2.21) is indeed harmonic in U.

Lemma 2.3. *Let U be a bounded domain in $\mathbb{R}^n$ and φ be a continuous function on ∂U. Then u_φ defined in (2.21) is harmonic in U.*

Proof. Set

$$\mathcal{S}_\varphi = \{v \,|\, v \in C(\bar{U}) \text{ is subharmonic in } U,\ v \leq \varphi \text{ on } \partial U\},$$

and we set $\mathcal{S} = \mathcal{S}_\varphi$ if there is no confusion in its meaning. Then for any $x \in U$,

$$u_\varphi(x) = \sup\{v(x) \,|\, v \in \mathcal{S}\}.$$

Step 1: The quanitity u_φ is well defined.

To show this, first set

$$m = \min_{\partial U} \varphi \ \text{ and } \ M = \max_{\partial U} \varphi.$$

We note that the constant function m is in $\mathcal{S}$ and thus the set $\mathcal{S}$ is non-empty. Next, the constant function M is clearly harmonic in U with $\varphi \leq M$ on ∂U. By Lemma 2.1, for any $v \in \mathcal{S}$,

$$v \leq M \ \text{ in } \bar{U}.$$

Thus u_φ is well-defined and $u_\varphi \leq M$ in U.

Step 2: We show $\mathcal{S}$ is closed by taking the maximum among finitely many functions in $\mathcal{S}$.

Choose arbitrary $v_1, v_2, \ldots, v_k \in \mathcal{S}$ and set

$$v = \max\{v_1, v_2, \ldots, v_k\}.$$

It follows easily, by definition, that v is subharmonic in U. Hence, $v \in \mathcal{S}$.

Step 3: We prove that u_φ is harmonic in any $B_r(x_0) \subset U$.

By definition of u_φ, there exists a sequence of functions $v_i \in \mathcal{S}$ such that

$$\lim_{i\to\infty} v_i(x_0) = u_\varphi(x_0).$$

We may replace v_i above by any $\tilde{v}_i \in \mathcal{S}$ with $\tilde{v}_i \geq v_i$ since

$$v_i(x_0) \leq \tilde{v}_i(x_0) \leq u_\varphi(x_0).$$

Replacing, if necessary, v_i by $\max\{m, v_i\} \in \mathcal{S}$, we may also assume

$$m \leq v_i \leq u_\varphi \text{ in } U.$$

For fix $B_r(x_0)$ and each v_i, we let w_i be the harmonic lifting in Lemma 2.2. Then $w_i = v_i$ in $U \backslash B_r(x_0)$ and

$$\begin{cases} \Delta w_i = 0 & \text{in } B_r(x_0), \\ w_i = v_i & \text{on } \partial B_r(x_0). \end{cases}$$

By Lemma 2.2, $w_i \in \mathcal{S}$ and $v_i \leq w_i$ in U. Moreover, w_i is harmonic in $B_r(x_0)$ and satisfies

$$\lim_{i\to\infty} w_i(x_0) = u_\varphi(x_0),$$
$$m \leq w_i \leq u_\varphi \text{ in } U,$$

for any $i = 1, 2, \ldots$. By the compactness of bounded harmonic functions (see Corollary 1.8), there exists a harmonic function w in $B_r(x_0)$ such that a subsequence of $\{w_i\}$, we still denote by $\{w_i\}$, converges to w on compact subsets of $B_r(x_0)$. We deduce that

$$w \leq u_\varphi \text{ in } B_r(x_0) \text{ and } w(x_0) = u_\varphi(x_0).$$

We now claim that $u_\varphi = w$ in $B_r(x_0)$. To see this, take any $\bar{x} \in B_r(x_0)$ and proceed similarly as before, with $\bar{x}$ replacing x_0. By definition of u_φ, there exists a sequence $\{\bar{v}_i\} \subset \mathcal{S}$ such that

$$\lim_{i\to\infty} \bar{v}_i(\bar{x}) = u_\varphi(\bar{x}).$$

As before, we can replace, if necessary, $\bar{v}_i$ by $\max\{\bar{v}_i, w_i\} \in \mathcal{S}$. So we may also assume that

$$w_i \leq \bar{v}_i \leq u_\varphi \text{ in } U.$$

For the fixed $B_r(x_0)$ and each $\bar{v}_i$, we let $\bar{w}_i$ be the harmonic lifting in Lemma 2.2. Then, $\bar{w}_i \in \mathcal{S}$ and $\bar{v}_i \leq \bar{w}_i$ in U. Moreover, $\bar{w}_i$ is harmonic in $B_r(x_0)$ and satisfies

$$\lim_{i\to\infty} \bar{w}_i(\bar{x}) = u_\varphi(\bar{x}),$$
$$m \leq \max\{\bar{v}_i, w_i\} \leq \bar{w}_i \leq u_\varphi \text{ in } U,$$

for any $i = 1, 2, \ldots$. Again, by compactness, there exists a harmonic function $\bar{w}$ in $B_r(x_0)$ with a maximum attained at x_0. Then, by the strong maximum principle applied to $w - \bar{w}$ in $B_{r'}(x_0)$ for any $r' < r$, we deduce that $w - \bar{w}$ is constant and thus is equal to zero. This implies $w = \bar{w}$ in $B_r(x_0)$ and particularly, $w(\bar{x}) = \bar{w}(\bar{x}) = u_\varphi(\bar{x})$. Hence, $w = u_\varphi$ in $B_r(x_0)$ since $\bar{x}$ was chosen arbitrarily in $B_r(x_0)$. This proves u_φ is harmonic in $B_r(x_0)$. □

Observe carefully that u_φ as given in the previous lemma is only defined in U. To discuss the limits of $u_\varphi(x)$ as x approaches the boundary, we must make some additional assumptions on the boundary of U, ∂U.

Lemma 2.4. *Let φ be a continuous function on ∂U and u_φ be the function defined in* (2.21)*. For some $x_0 \in \partial U$, suppose $w_{x_0} \in C(\bar{U})$ is a subharmonic function in U such that*

$$w_{x_0}(x_0) = 0, \quad w_{x_0}(x) < 0 \ \text{ for any } \ x \in \partial U \backslash \{x_0\}, \tag{2.22}$$

then

$$\lim_{x \to x_0} u_\varphi(x) = \varphi(x_0).$$

Proof. As before, consider the set

$$\mathcal{S}_\varphi = \{v \,|\, v \in C(\bar{U}) \ \text{ is subharmonic in } U, \, v \leq \varphi \ \text{ on } \ \partial U\}.$$

To simplify notation, we just write $w = w_{x_0}$ and set $M = \max_{\partial U} |\varphi|$. Let $\varepsilon > 0$ be arbitrary, and by the continuity of φ at x_0, there exists a $\delta > 0$ such that

$$|\varphi(x) - \varphi(x_0)| < \varepsilon \ \text{ for any } \ x \in \partial U \cap B_\delta(x_0).$$

We then choose K suitably large so that $-Kw(x) \geq 2M$ for any $x \in \partial U \backslash B_\delta(x_0)$. Thus,

$$|\varphi(x) - \varphi(x_0)| < \varepsilon - Kw \ \text{ for } \ x \in \partial U.$$

Since $\varphi(x_0) - \varepsilon + Kw(x)$ is a subharmonic function in U with $\varphi(x_0) - \varepsilon + Kw \leq \varphi$ on ∂U, we have that $\varphi(x_0) - \varepsilon + Kw \in \mathcal{S}_\varphi$. The definition of u_φ then implies that

$$\varphi(x_0) - \varepsilon + Kw \leq u_\varphi \ \text{ in } \ U. \tag{2.23}$$

However, $\varphi(x_0) + \varepsilon - Kw$ is super-harmonic in U with $\varphi(x_0) + \varepsilon - Kw \geq \varphi$ on ∂U. Thus, for any $v \in \mathcal{S}_\varphi$, we obtain from Lemma 2.1

$$v(x) \leq \varphi(x_0) + \varepsilon - Kw(x) \ \text{ for } \ x \in U.$$

Again, by the definition of u_φ,

$$u_\varphi(x) \leq \varphi(x_0) + \varepsilon - Kw(x) \ \text{ for } \ x \in U. \tag{2.24}$$

Hence, (2.23) and (2.24) imply

$$|u_\varphi(x) - \varphi(x_0)| < \varepsilon - Kw(x) \text{ for } x \in U,$$

and since w is continuous so that $w(x) \longrightarrow w(x_0) = 0$ as $x \longrightarrow x_0$, we arrive at

$$\limsup_{x \to x_0} |u_\varphi(x) - \varphi(x_0)| < \varepsilon.$$

The desired result follows once after sending $\varepsilon \longrightarrow 0$.

□

Remark 2.9. *The function w_{x_0} satisfying* (2.22) *is often called a barrier function. Barrier functions can be constructed for a large class of domains. One type of domain, for instance, is when U satisfies an* **exterior sphere condition** *at $x_0 \in \partial U$, i.e., there exists a ball $B_{r_0}(y_0)$ such that*

$$U \cap B_{r_0}(y_0) = \emptyset, \quad \bar{U} \cap \bar{B}_{r_0}(y_0) = \{x_0\}.$$

To construct a barrier function at x_0, we take

$$w_{x_0}(x) = \Gamma(x - y_0) - \Gamma(x_0 - y_0) \text{ for any } x \in \bar{U}$$

where Γ is the fundamental solution of Laplace's equation. Therefore, w_{x_0} is harmonic in U and satisfies (2.22)*. In addition, we mention that the exterior sphere condition always holds for C^2 domains.*

Combining the previous lemmas and remark, we have essentially constructed a solution $u = u_\varphi$ to the Dirichlet problem (2.20). That is, we have shown the following existence result.

Theorem 2.17. *Let U be a bounded domain in $\mathbb{R}^n$ satisfying the exterior sphere condition at every boundary point. Then, for any $\varphi \in C(\partial U)$, the Dirichlet problem* (2.20) *admits a solution $u \in C^\infty(U) \cap C(\bar{U})$.*

In summary, the solvability of the Dirichlet problem for Laplace's equation depends on both the data g and the geometry of the domain U. As indicated in Lemma 2.4, the issue revolves around the following question. When can the harmonic function from the Perron method be extended continuously up to the boundary? In other words, when are the points of the boundary regular with respect to the Laplacian? Of course, g being continuous on ∂U and U satisfying the exterior sphere condition are enough to give a positive answer to this question. Alternatively, another criterion indicating when a boundary point is regular with respect to the Laplacian can be given in terms of 2-capacities. This criterion is called the Wiener criterion, and it can be generalized to uniformly elliptic equations in divergence form.

Let $n \geq 3$ and

$$K^p = \{f : \mathbb{R}^n \to \mathbb{R}_+ \mid f \in L^{p^*}(\mathbb{R}^n), Df \in L^p(\mathbb{R}^n;\mathbb{R}^n)\}.$$

If $A \subset \mathbb{R}^n$, we define the p-capacity of A by

$$Cap_p(A) = \inf\Big\{\int_{\mathbb{R}^n} |Df|^p\, dx \,:\, f \in K^p, A \subseteq interior\{f \geq 1\}\Big\}. \tag{2.25}$$

By regularization, note that

$$Cap_p(K) = \inf\Big\{\int_{\mathbb{R}^n} |Df|^p\, dx \,:\, f \in C_c^\infty(\mathbb{R}^n),\ f \geq \chi_K\Big\}$$

for each compact set $K \subset \mathbb{R}^n$.

Let x_0 be a boundary point in ∂U. Then for any fixed $\lambda \in (0,1)$, let

$$A_j = \{x \notin U \,:\, |x - x_0| \leq \lambda^j\}.$$

The **Wiener criterion** states that x_0 is a regular boundary point of U if and only if the series

$$\sum_{j=0}^{\infty} \frac{Cap_2(A_j)}{\lambda^{j(n-2)}}$$

diverges.

2.6 Continuity Method

In this section, we introduce the continuity method to prove the existence of classical solutions to general uniformly elliptic equations of second-order. One crucial ingredient of the method relies on global $C^{2,\alpha}$ a priori estimates of solutions (see the Schauder estimates in Section 3.5) and this provides one important application of the regularity theory for such equations. In the next chapter, we will investigate the various types of regularity properties of solutions to uniformly elliptic equations in great detail.

Let $U \subset \mathbb{R}^n$ be a bounded domain, let a^{ij}, b^i and c be defined in U with a^{ij} symmetric. Consider the second-order elliptic operator

$$Lu = -a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u \text{ in } U$$

and assume L is uniformly elliptic in the following sense:

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for any } x \in U \text{ and } \xi \in \mathbb{R}^n$$

for some positive constant $\lambda > 0$.

We prove the following general existence result for solutions of Dirichlet boundary value problem with $C^{2,\alpha}$ boundary values involving the operator L with C^α coefficients.

Theorem 2.18. *Let $U \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain, let L be a uniformly elliptic operator as defined as above with $c \geq 0$ in U and $a^{ij}, b, c \in C^\alpha(U)$ for some $\alpha \in (0,1)$. Then for any $f \in C^\alpha(\bar{U})$ and $\varphi \in C^{2,\alpha}(\bar{U})$, there exists a unique solution $u \in C^{2,\alpha}(\bar{U})$ of the Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } U, \\ u = \varphi & \text{on } \partial U. \end{cases} \tag{2.26}$$

In fact, we shall prove the solvability of the boundary value problem (2.26) if the same is true for the boundary value problem with $L = -\Delta$, i.e., for Poisson's equation. Of course, the latter is a basic known result and so Theorem 2.18 follows accordingly.

Theorem 2.19. *Let $U \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain, let L be a uniformly elliptic operator as defined above with $c \geq 0$ in U and $a^{ij}, b, c \in C^\alpha(U)$ for some $\alpha \in (0,1)$. If the Dirichlet problem for Poisson's equation*

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = \varphi & \text{on } \partial U, \end{cases} \tag{2.27}$$

has a $C^{2,\alpha}(\bar{U})$ solution for all $f \in C^\alpha(\bar{U})$ and $\varphi \in C^{2,\alpha}(\bar{U})$, then the Dirichlet problem,

$$\begin{cases} Lu = f & \text{in } U, \\ u = \varphi & \text{on } \partial U, \end{cases} \tag{2.28}$$

also has a (unique) $C^{2,\alpha}(\bar{U})$ solution for all such f and φ.

Proof. Without loss of generality, we assume $\varphi \equiv 0$; otherwise, we consider $Lv = f - L\varphi$ in U and $v = 0$ on ∂U.

Consider the family of equations:

$$L_t u \equiv tLu + (1-t)(-\Delta)u = f$$

for $t \in [0,1]$. We note that $L_0 = -\Delta$ and $L_1 = L$.

If we write

$$L_t u = a_t^{ij}(x)D_{ij}u + b_t^i(x)D_i u + c_t(x)u,$$

we can easily verify that

$$a_t^{ij}(x)\xi_i\xi_j \geq \min(1,\lambda)|\xi|^2$$

for any $x \in U$ and $\xi \in \mathbb{R}^n$ and that

$$|a_t^{ij}|_{C^\alpha(\bar{U})}, |b_t^i|_{C^\alpha(\bar{U})}, |c_t|_{C^\alpha(\bar{U})} \leq \max(1,\Lambda)$$

independently of $t \in [0,1]$. Thus,

$$|L_t u|_{C^\alpha(\bar{U})} \leq C|u|_{C^{2,\alpha}(U)}$$

where C is a positive constant depending only on $n, \alpha, \lambda, \Lambda$ and U. Then for each $t \in [0, 1]$, $L_t : X \to C^\alpha(U)$ is a bounded operator, where

$$X = \{u \in C^{2,\alpha}(\bar{U}) \,|\, u = 0 \text{ on } \partial U\}$$

is the Banach space equipped with the norm $|\cdot|_{C^{2,\alpha}(\bar{U})}$.

Define the set I containing the points $s \in [0, 1]$ such that the Dirichlet problem

$$\begin{cases} L_s u = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2.29}$$

is solvable in $C^{2,\alpha}(\bar{U})$ for any $f \in C^\alpha(\bar{U})$. We take an $s \in I$ and let $u = L_s^{-1} f$ be the (unique) solution. Then, standard global $C^{2,\alpha}$ estimates (cf. Theorem 1.38) and the maximum principle imply

$$|L_s^{-1} f|_{C^{2,\alpha}(U)} \leq C|f|_{C^\alpha(\bar{U})}.$$

For any $t \in [0, 1]$ and $f \in C^\alpha(\bar{U})$, we can write $L_t u = f$ as

$$L_s u = f + (L_s - L_t)u = f + (t - s)(\Delta u - Lu).$$

Hence, $u \in C^{2,\alpha}(\bar{U})$ is a solution of

$$\begin{cases} L_t u = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

if and only if

$$u = L_s^{-1}(f + (t - s)(\Delta - L)u).$$

For any $u \in X$, set

$$Tu = L_s^{-1}(f + (t - s)(\Delta u - Lu))$$

so that $T : X \longrightarrow X$ is an operator, and we claim T is a contraction mapping. Indeed, for any $u, v \in X$,

$$\begin{aligned} |Tu - Tv|_{C^{2,\alpha}(\bar{U})} &= |(t - s)L_s^{-1}((\Delta - L)(u - v))|_{C^{2,\alpha}(\bar{U})} \\ &\leq C|t - s||(\Delta - L)(u - v)|_{C^\alpha(\bar{U})} \leq C|t - s||u - v|_{C^{2,\alpha}(\bar{U})}. \end{aligned}$$

Therefore, $T : X \to X$ is a contraction mapping if $|t-s| < \delta := C^{-1}$. Hence, for any $t \in [0, 1]$ with $|t - s| < \delta$, there exists a unique $u \in X$ such that $u = Tu$, i.e.,

$$u = L_s^{-1}(f + (t - s)(\Delta u - Lu)).$$

Namely, for any $t \in [0, 1]$ with $|t - s| < \delta$ and any $f \in C^\alpha(\bar{U})$, there exists a solution of $u \in C^{2,\alpha}(\bar{U})$ of

$$\begin{cases} L_t u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Therefore, if $s \in I$, then $t \in I$ for any $t \in [0, 1]$ with $|t - s| < \delta$. So we can divide the interval $[0, 1]$ into subintervals of length less than δ. By $0 \in I$, we deduce $1 \in I$. This completes the proof of the theorem. □

2.7 Unique Solvability of the Dirichlet Problem via Schauder Estimates and Maximum Principles

In this section, we establish existence and uniqueness of a classical solution to the following general Dirichlet problem,

$$\begin{cases} Lu = f & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases} \tag{2.30}$$

where

$$Lu = -\sum_{i,j=1}^{n} D_j\left(a^{ij}(x)D_i u\right) + \sum_{j=1}^{n} b^i(x)D_i u + c(x)u.$$

We suppose $a^{ij}, b^i, c \in C^{\alpha}(\bar{U})$, and L is uniformly elliptic, i.e., there exist $0 < \lambda \leq \Lambda$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for all } x \in U,\ \xi \in \mathbb{R}^n.$$

We also assume

$$\frac{1}{\lambda}\Big\{\sum_{i,j=1}^{n} |a^{ij}|_{\alpha;U} + \sum_{i=1}^{n} |b^i|_{\alpha;U} + |c|_{\alpha;U}\Big\} \leq \Lambda_{\alpha}.$$

If our domain and its boundary are nice enough, we derive our existence result using a simple iteration argument.

Proposition 2.1. *Let $\partial U \in C^{[n/2]+4}$, and suppose $c \geq 0$, $f \in C^{\alpha}(\bar{U})$ and $g \in C^{2,\alpha}(\bar{U})$, where $0 < \alpha < 1$. Then the Dirichlet problem (2.30) admits a unique solution u belonging to $C^{2,\alpha}(\bar{U})$.*

Proof. Assume, without loss of generality, that $g \equiv 0$. And for all $1 \leq i, j \leq n$, suppose the sequences a_k^{ij}, b_k^i, c_k, and f_k are in the class $C^{\infty}(\bar{U})$ and converge uniformly on $\bar{U}$, respectively, to a^{ij}, b^i, c and f with $c_k \geq 0$, $\|f_k\|_{\alpha;U} \leq 2\|f\|_{\alpha;U}$,

$$\frac{\lambda}{2}|\xi|^2 \leq \sum_{i,j=1}^{n} a_k^{ij}\xi_i\xi_j \leq 2\Lambda|\xi|^2 \quad \text{for all } x \in U,\ \xi \in \mathbb{R}^n,$$

and

$$\frac{1}{\lambda}\Big\{\sum_{i,j=1}^{n} |a_k^{ij}|_{\alpha} + \sum_{i=1}^{n} |b_k^i|_{\alpha} + |c_k|_{\alpha}\Big\} \leq 2\Lambda_{\alpha}.$$

Then consider the sequence of approximate Dirichlet problems,

$$\begin{cases} L_k u^k = f_k & \text{in } U, \\ u^k = g = 0 & \text{on } \partial U, \end{cases} \tag{2.31}$$

where

$$L_k u = -\sum_{i,j=1}^{n} D_j \left(a_k^{ij}(x) D_i u\right) + \sum_{j=1}^{n} b_k^i(x) D_i u + c_k(x) u. \tag{2.32}$$

For each $k = 1, 2, 3, \ldots$, the existence of a unique weak solution $u^k \in H_0^1(U)$ of (2.1) can be deduced from the previous sections. In fact, the H^k-regularity theory for weak solutions (see, e.g., Theorem 3.16) implies that $u^k \in H^{[n/2]+4}(U) \cap H_0^1(U)$ and thus $u^k \in C^{2,\alpha}(\bar{U})$ thanks to the Sobolev embedding theorem. Then, by the Global Schauder estimates of Theorem 1.38,

$$|u^k|_{2,\alpha,U} \leq C|f_k|_{\alpha;U} \leq 2C|f|_{\alpha;U},$$

where $C > 0$ is independent of k. Thus, by the Arzelà-Ascoli Theorem, there exists a subsequence of solutions, which we still denote by $\{u^k\}$, that converges to some function $u \in C^2(\bar{U})$ that satisfies (2.30). The $C^{2,\alpha}$ regularity of the limit solution u is a consequence of the global Schauder estimates, and its uniqueness is ensured by the strong maximum principle. □

The previous result imposes a fairly strong regularity assumption on the boundary ∂U. We can remove this so long as we assume our domain satisfies the exterior sphere condition. Specifically, the idea is to find a unique limiting solution just as we did in the previous proposition. Then, the exterior sphere condition ensures this limiting solution satisfies the boundary condition, and to verify this, we shall make use of the maximum principle combined with a carefully chosen barrier function. The approach is similar to what we looked at earlier with the Perron method.

Theorem 2.20. *Let U satisfy the exterior sphere condition defined in 2.9, and besides the strong regularity condition on the boundary, we assume the same conditions as given in Proposition 2.1. Then the Dirichlet problem* (2.30) *admits a unique solution u belonging to $C^{2,\alpha}(U) \cap C(\overline{U})$.*

Proof. **Step 1:** To invoke Proposition 2.1, we extract a sequence of domains $U_k \subset U$ such that $\partial U_k \in C^{[n/2]+4}$ and $\sup_{x \in \partial U_k} dist(x, \partial U) < 1/k$. We also choose a sequence $g_k \in C^{2,\alpha}(\bar{U})$ such that $|g_k - g|_{0;U} \leq 1/k$. By Proposition 2.1, the class of Dirichlet problems,

$$\begin{cases} L_k u^k = f & \text{in } U, \\ u^k = g_k & \text{on } \partial U, \end{cases} \tag{2.33}$$

where L_k is defined as in (2.32), admits a unique solution $u^k \in C^{2,\alpha}(\bar{U})$ for each $k = 1, 2, 3, \ldots$. For any $U' \subset\subset U$, applying the interior Schauder estimate (Theorem 3.20) followed by the weak maximum principle (Theorem 1.12) yields, for sufficiently large k,

$$|u^k|_{2,\alpha;U'} \leq C(|f|_{\alpha;U} + |u^k|_{0;U}) \leq C(|f|_{\alpha;U} + 1/k),$$

where $C > 0$ is independent of k. The Arzelá-Ascoli theorem further ensures we can extract a subsequence, which we still denote by $\{u^k\}$, that converges in $C^2(\bar{U}')$ to some function

$u \in C^{2,\alpha}(\bar{U}')$, for any $U' \subset\subset U$. By construction of this sequence of solutions to (2.33), this function u must satisfy the equation $Lu = f$ in U.

Step 2: It remains to show the limit function satisfies $u = g$ on ∂U.

Obviously, $u \in C(\bar{U})$ and we show $\lim_{x \longrightarrow x^0} u(x) = g(x^0)$ for each boundary point $x^0 \in \partial U$. So pick any $x^0 \in \partial U$. Consider the function

$$\omega(x) = e^{-\beta\rho^2} - e^{-\beta|x-y|^2}$$

where $\beta > 0$ is a constant to be determined later, and $B_\rho(y)$ is the exterior ball as guaranteed by the exterior sphere condition. Note that $\omega(x)$ is a barrier function because it satisfies the following,

1. $\omega(x^0) = 0$ and $\omega > 0$ in $\bar{U} \backslash \{x^0\}$;
2. $\omega \in C^2(\bar{U})$ and $L\omega > 0$.

As $c \geq 0$ and if we fix β sufficiently large enough, we obtain $L\omega \geq c > 0$ for some constant c. Now, by continuity, for each $\epsilon > 0$ there exists a ball $B_\delta(x^0)$ such that

$$|g(x) - g(x^0)| < \epsilon \text{ for all } x \in B_\delta(x^0) \cap U.$$

Since ω is bounded from below away from zero on $U \backslash B_\delta(x^0)$, we can find a large enough constant $C > 0$, depending on ϵ, such that

$$-C\omega(x) + g(x^0) - \epsilon < g(x) < C\omega(x) + g(x^0) + \epsilon \text{ for all } x \in \bar{U},$$

so for all large enough k,

$$-C\omega(x) + g(x^0) - \epsilon \leq g_k(x) \leq C\omega(x) + g(x^0) + \epsilon \text{ for all } x \in \bar{U}.$$

From $L\omega \geq c > 0$, for suitably large constant $C > 0$, we get

$$L(-C\omega(x) + g(x^0) - \epsilon) \leq Lu^k(x) \leq L(C\omega(x) + g(x^0) + \epsilon) \text{ for all } x \in U_k.$$

Thus, the weak maximum principle implies

$$-C\omega(x) + g(x^0) - \epsilon \leq u^k(x) \leq C\omega(x) + g(x^0) + \epsilon \text{ for all } x \in U_k.$$

Sending $k \longrightarrow \infty$ yields

$$-C\omega(x) + g(x^0) - \epsilon \leq u(x) \leq C\omega(x) + g(x^0) + \epsilon \text{ for all } x \in U.$$

Therefore

$$g(x^0) - \epsilon \leq \liminf_{x \longrightarrow x^0} u(x) \leq \limsup_{x \longrightarrow x^0} u(x) \leq g(x^0) + \epsilon$$

and thus

$$\lim_{x \longrightarrow x^0} u(x) = g(x^0)$$

and because x^0 was chosen arbitrarily, this confirms $u = g$ on ∂U thereby finishing the proof. □

2.8 Sub-Supersolution methods

Let U be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^n$ with $0 < \alpha \leq 1$ and suppose f is a C^1 function in $\overline{U} \times \mathbb{R}$. The following technique constructs a sequence converging to a desired $C^{2,\alpha}(\overline{U})$ solution of the semilinear problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2.34}$$

provided we can find a subsolution $\underline{u}$ and a supersolution $\overline{u}$. That is, $\underline{u}, \overline{u} \in C^{2,\alpha}(\overline{U})$ such that $-\Delta \underline{u} \leq f(x,\underline{u})$ in U, $\underline{u} \leq 0$ on ∂U, and $-\Delta \overline{u} \geq f(x,\overline{u})$ in U, $\overline{u} \geq 0$ on ∂U. We then invoke compactness properties of the Hölder spaces, which requires we make use of the classical Schauder regularity estimates. We shall only reference the regularity estimates required here for the sole purpose of establishing our existence results; however, a detailed examination of regularity theorems along with their proofs will be provided in the next chapter.

Theorem 2.21. *Let $\underline{u}$ (and, resp., $\overline{u}$) belong to $C^{2,\alpha}(\overline{U})$ and is a subsolution (and, resp., supersolution) of* (2.34) *such that $\underline{u} \leq \overline{u}$. Then there exists a solution $u \in C^{2,\alpha}(\overline{U})$ of* (2.34) *such that $\underline{u} \leq u \leq \overline{u}$ in U*

Proof. Set

$$m = \inf_U \underline{u} \quad \text{and} \quad M = \sup_U \overline{u}.$$

Pick $\lambda > 0$ large enough so that

$$\lambda > -\partial_z f(x,z)$$

for any $(x,z) \in \overline{U} \times [m,M]$.

Our strategy is to construct a compact sequence $\{u_k\}_{k=1}^{\infty}$ whose limit is the desired solution.

Step 1: We start by letting $u_0 = \underline{u}$. For any u_k, $k = 0,1,2,\ldots$, we suppose $u_{k+1} \in C^{2,\alpha}(\overline{U})$ is a solution of

$$\begin{cases} -\Delta u_{k+1} + \lambda u_{k+1} = f(x,u_k) + \lambda u_k & \text{in } U, \\ u_{k+1} = 0 & \text{on } \partial U. \end{cases} \tag{2.35}$$

We claim for all k,

$$\underline{u} \leq u_k \leq \overline{u} \ \text{ in } U. \tag{2.36}$$

This is obviously true for $k = 0$. Proceeding by induction, suppose the claim holds for some integer $k \geq 0$. Observing that

$$-\Delta(u_{k+1} - \underline{u}) + \lambda(u_{k+1} - \underline{u}) \geq (f(x,u_k) - f(x,\underline{u})) + \lambda(u_k - \underline{u}),$$

so by the mean-value theorem, there holds

$$f(x,u_k) - f(x,\underline{u}) + \lambda(u_k - \underline{u}) = (\partial_z f(x,\bar{z}) + \lambda)(u_k - \underline{u})$$

for some value $\bar{z}$ between $u_k(x)$ and $\underline{u}(x)$. Hence

$$\begin{cases} -\Delta(u_{k+1} - \underline{u}) + \lambda(u_{k+1} - \underline{u}) \geq 0 & \text{in } U, \\ u_{k+1} - \underline{u} \geq 0 & \text{on } \partial U. \end{cases} \tag{2.37}$$

Applying the maximum principle yields $u_{k+1} \geq \underline{u}$ in U. Likewise, similar arguments will lead to $u_{k+1} \leq \overline{u}$ in U. That is,

$$m \leq u_k(x) \leq M \quad \text{for any } x \in U \text{ and } k = 0, 1, 2, \ldots.$$

Step 2: We also prove that

$$\underline{u} \leq u_1 \leq u_2 \leq u_3 \leq \ldots \leq \overline{u}.$$

The argument basically mirrors the previous one. Namely, we assume

$$\underline{u} \leq u_1 \leq u_2 \leq \ldots u_{k-1} \leq u_k \leq \overline{u}$$

for some integer $k \geq 0$. In view of (2.35) and the mean-value theorem, we get

$$\begin{cases} -\Delta(u_{k+1} - u_k) + \lambda(u_{k+1} - u_k) \geq 0 & \text{in } U, \\ u_{k+1} - u_k \geq 0 & \text{on } \partial U. \end{cases} \tag{2.38}$$

The maximum principle implies that $u_{k+1} - u_k \geq 0$ in U and this proves the claim.

Step 3: From the monotonicity and boundedness of $\{u_k\}_{k=0}^{\infty}$, there exists a function u in U such that $u_k(x) \longrightarrow u(x)$ for each $x \in U$. Noting that the right-hand side of the PDE in (2.35) is uniformly bounded independent of k, the global $C^{1,\alpha}$ Schauder estimate (see Section 3.5) implies that $\|u_k\|_{C^{1,\alpha}(\overline{U})} \leq C$ for some constant $C > 0$ depending only on n, λ, m, M and U and therefore independent of k. Hence, that same right-hand side of (2.35) is uniformly bounded in the $C^{1,\alpha}$ norm independent of k. The global $C^{2,\alpha}$ Schauder estimates (see Section 3.5) reveal $\|u_k\|_{C^{2,\alpha}(\overline{U})} \leq C$ for some constant $C > 0$ depending only on n, λ, m, M and U. By the Arzela-Ascoli theorem, we may extract a subsequence u_{k_j} of u_k such that $u_{k_j} \longrightarrow u$ in $C^2(U)$. This limit function u is a solution of (2.34) that satisfies the desired properties. □

2.9 Calculus of Variations I: Minimizers and Weak Solutions

Another important approach to establishing the existence of weak solutions to elliptic equations is through variational methods. This is especially important since if we are searching for weak solutions of semilinear equations, $Lu = f(x, u)$, then the Lax–Milgram theorem no longer applies. Variational methods are often used to circumvent this issue. The key idea is to carefully identify an associated energy functional of the elliptic equation whose critical points are indeed weak solutions of the elliptic problem.

Remark 2.10. *Although variational methods are used to find weak solutions, elliptic regularity theory often ensures that weak solutions are actually strong or classical solutions.*

We begin with a simple example for the sake of illustration. Consider

$$\begin{cases} -\Delta u = f(x) & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2.39}$$

and consider the functional

$$J(u) = \frac{1}{2}\int_U |Du|^2\,dx - \int_U f(x)u\,dx, \;\; u \in H_0^1(U). \tag{2.40}$$

Remark 2.11. *In general, we will consider the semilinear case when $f = f(x,u)$ case. In the special case where $f(x,u) = |u|^{p-1}u$, then we get the problem*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \textit{in } U, \\ u = 0 & \textit{on } \partial U. \end{cases} \tag{2.41}$$

Equation (2.41) *is often called the Lane-Emden equation. It serves as the prototypical semilinear equation, and it is the model that we will study in great detail throughout these notes. Indeed, the exponent p has important implications in both the quantitative and qualitative properties of solutions and there are three primary cases to consider. In particular, we say the equation is subcritical, critical or super-critical, respectively, if $p < \frac{n+2}{n-2}$, $p = \frac{n+2}{n-2}$ or $p > \frac{n+2}{n-2}$.*

We now show that if u is a minimizer of this functional $J(\cdot)$ in the class of $H_0^1(U)$, then u a weak solution of (2.39). Let v be any function in $H_0^1(U)$ and consider the real-valued function

$$g(t) = J(u+tv), \;\; t \in \mathbb{R}.$$

Since u is a minimizer of $J(\cdot)$, the function $g(t)$ has a minimum at $t = 0$, and thus we must have

$$0 = g'(0) = \frac{d}{dt}J(u+tv)\Big|_{t=0},$$

where explicitly,

$$J(u+tv) = \frac{1}{2}\int_U |D(u+tv)|^2\,dx - \int_U f(x)(u+tv)\,dx,$$

and

$$\frac{d}{dt}J(u+tv) = \int_U D(u+tv)\cdot Dv\,dx - \int_U f(x)v\,dx.$$

Hence, $g'(0) = 0$ implies

$$\int_U Du\cdot Dv\,dx - \int_U f(x)v\,dx = 0, \;\; \text{for all } \; v \in H_0^1(U),$$

and so u is a weak solution of (2.39).

Remark 2.12. *The first derivative $g'(0)$ is often called the* **first variation** *of $J(\cdot)$. In the next chapter, we develop the regularity theory for the weak solutions of such elliptic problems. In particular, it follows that the weak solution of* (2.39) *obtained by our variational method is a classical solution provided f is regular enough, e.g., it is Hölder continuous.*

Clearly, for u to be a weak solution it need not be a minimum; it can be a maximum or saddle point of the functional, or generally any point that satisfies

$$0 = \frac{d}{dt} J(u+tv)\Big|_{t=0}.$$

This motivates the following definition.

Definition 2.6. *Let $J = J(\cdot)$ be a functional on a Banach space X.*

(a) We say that J is Frechet differentiable at $u \in X$ if there exists a continuous linear map $\mathcal{L} : X \longrightarrow X^*$ satisfying: For any $\epsilon > 0$, there is a $\delta = \delta(\epsilon, u)$ such that

$$|J(u+v) - J(u) - \langle \mathcal{L}(u), v\rangle| \le \epsilon \|v\|_X \;\; \textit{whenever} \;\; \|v\|_X < \delta.$$

The mapping $\mathcal{L}(u)$ is commonly denoted by $J'(u)$.

(b) A critical point of J is a point at which $J'(u) = 0$; that is,

$$\langle J'(u), v\rangle = 0 \;\; \textit{for all} \;\; v \in X.$$

We call $J'(u) = 0$, and the PDE associated with this distribution equation, the **Euler-Lagrange** *equation of the functional $J(\cdot)$.*

Remark 2.13. *One can verify that if J is Frechet differentiable at u, then*

$$\langle J'(u), v\rangle = \lim_{t \longrightarrow 0} \frac{J(u+tv) - J(u)}{t} = \frac{d}{dt} J(u+tv)\Big|_{t=0}.$$

More generally, given the **Lagrangian** $L : \mathbb{R}^n \times \mathbb{R} \times \bar{U} \longrightarrow \mathbb{R}$ with $L = L(p, z, x)$ and using the notation

$$\begin{cases} D_p L = (L_{p_1}, \dots, L_{p_n}), \\ D_z L = L_z, \\ D_x L = (L_{x_1}, \dots, L_{x_n}), \end{cases}$$

we may consider the functional

$$J(u) = \int_U L(Du(x), u(x), x)\, dx.$$

As before, we may compute the **Euler-Langrange** equation associated with this functional $J(\cdot)$ to be the divergence-form elliptic equation

$$-\sum_{i=1}^n (L_{p_i}(Du, u, x))_{x_i} + L_z(Du, u, x) = 0 \;\; \text{in } U.$$

Although we will mostly focus on the special case

$$L(p,z,x)=\frac{1}{2}|p|^2-zf(x), \tag{2.42}$$

which corresponds to the functional (2.40), the results we cover extend to more general Lagrangians under some coercivity and convexity assumptions on L (see Chapter 8 in [9]). We state these general conditions and the accompanying variational existence results at the end of the next subsection. There we will first consider the model example (2.42) and provide the proof of the main existence result in this case. We shall omit the proof for the general case though they are very similar.

2.9.1 Existence of Weak Solutions

We prove the following theorem.

Theorem 2.22. *Suppose that $U\subset\mathbb{R}^n$ is a bounded domain with smooth boundary ∂U. Then for every $f\in L^{\frac{2n}{n+2}}(U)$ with $n>2$, the functional*

$$J(u)=\frac{1}{2}\int_U|Du|^2\,dx-\int_U f(x)u\,dx$$

possesses a minimum $u_0\in H_0^1(U)$, which is a weak solution of the boundary value problem

$$\begin{cases}-\Delta u=f(x) & \text{in } U,\\ \quad\; u=0 & \text{on } \partial U.\end{cases} \tag{2.43}$$

Proof. Let u_k be a minimizing sequence, i.e.,

$$\inf_{u\in H_0^1(U)}J(u)=\lim_{k\longrightarrow\infty}J(u_k).$$

Our goal is to show there does exist a function $u_0\in H_0^1(U)$ such that

$$J(u_0)=\lim_{k\longrightarrow\infty}J(u_k)=\inf_{u\in H_0^1(U)}J(u),$$

and as discussed earlier, u_0 is indeed a weak solution of the boundary value problem (2.43). To prove the existence of a minimum of the functional J, there are three main ingredients to verify: the functional J is

1. bounded from below,

2. coercive, and

3. weakly lower semi-continuous on $H_0^1(U)$.

1. We prove that J is bounded from below in $H := H_0^1(U)$ if $f \in L^2(U)$. From Poincaré's inequality, we endow the following equivalent norm on H:

$$\|u\|_H := \left(\int_U |Du|^2\,dx\right)^{1/2}$$

Thus, by Hölder and Poincaré's inequalities, we have

$$J(u) \geq \frac{1}{2}\|u\|_H^2 - C\|u\|_H\|f\|_{L^2(U)} = \frac{1}{2}\left(\|u\|_H - C\|f\|_{L^2(U)}\right)^2 - \frac{C^2}{2}\|f\|_{L^2(U)}^2 \geq -\frac{C^2}{2}\|f\|_{L^2(U)}^2.$$

2. Observe that a function bounded below does not guarantee it has a minimum. Take, for instance, $\frac{1}{1+x^2}$ on the real line. For a given minimizing sequence, we must make certain that the sequence does not "leak" to infinity. This motivates our need for a **coercive** condition. That is, if a sequence $\{u_k\}$ tends to infinity, i.e., $\|u_k\|_H \longrightarrow \infty$, then $J(u_k)$ must also become unbounded. In fact, it is clear that $J(u_k) \longrightarrow \infty$ as $\|u_k\|_H \longrightarrow \infty$ for our specific problem. This implies that a minimizing sequence would be retained in a bounded set; that is, any minimizing sequence must be bounded in H.

By the reflexivity of the Hilbert space H and the weak-$*$ compactness of the unit ball, the minimizing sequence has a weakly convergent subsequence, we still denote $\{u_k\}$, in H with limit point $u_0 \in H$. We shall show that u_0 is a minimum point of J.

3. We prove J is weakly lower semi-continuous on H.

Definition 2.7. *We say a functional $J(\cdot)$ is* **weakly lower semi-continuous** *on a Banach space X if for every weakly convergent sequence*

$$u_k \rightharpoonup u_0 \ \ in \ \ X,$$

we have

$$J(u_0) \leq \liminf_{k\longrightarrow\infty} J(u_k).$$

Clearly, it holds from the definition that $J(u_0) \geq \liminf_{k\longrightarrow\infty} J(u_k)$. Thus, if J is weakly lower semi-continuous, then $J(u_0) = \lim_{k\longrightarrow\infty} J(u_k)$. Hence, u_0 is a minimum of J and this completes the proof of the theorem provided we show J is weakly lower semi-continuous on H. Note that since $f \in L^{\frac{2n}{n+2}}(U)$, Hölder's inequality implies that $u \longrightarrow \int_U f(x)u\,dx$ is a continuous linear functional on H and thus,

$$\int_U f(x)u_k\,dx \longrightarrow \int_U f(x)u_0\,dx \ \text{ as } \ k \longrightarrow \infty. \tag{2.44}$$

From the algebraic inequality $a^2 + b^2 \geq 2ab$, we get $|Du_k|^2 + |Du_0|^2 \geq 2Du_0 \cdot Du_k$ or

$$\int_U |Du_k|^2\,dx + \int_U |Du_0|^2\,dx \geq 2\int_U Du_0 \cdot Du_k\,dx,$$

which after subtracting $2\int_U |Du_0|^2\,dx$ on both sides of this inequality yields

$$\int_U |Du_k|^2\,dx \geq \int_U |Du_0|^2\,dx + 2\int_U Du_0\cdot(Du_k - Du_0)\,dx.$$

This leads to

$$\liminf_{k\longrightarrow\infty} \int_U |Du_k|^2\,dx \geq \int_U |Du_0|^2\,dx,$$

since

$$\int_U Du_0\cdot(Du_k - Du_0)\,dx \longrightarrow 0 \ \text{ as } \ k \longrightarrow \infty.$$

Combining this with (2.44) yields the desired result. □

In general, assume $1 < q < \infty$ is fixed, choose some prescribed boundary condition g and define the admissible set

$$\mathcal{A} = \{w \in W^{1,q}(U)\,|\,w = g \text{ on } \partial U \text{ in the trace sense}\},$$

and suppose $L : \mathbb{R}^n \times \mathbb{R} \times \bar{U} \longrightarrow \mathbb{R}$ with $L = L(p,z,x)$ satisfies the following coercivity condition:

There exist constants $\alpha > 0$, $\beta \geq 0$ such that

$$L(p,z,x) \geq \alpha|p|^q - \beta \ \text{ for all } \ p \in \mathbb{R}^n,\ z \in \mathbb{R},\ x \in U. \tag{2.45}$$

This condition ensures that $J[w] \geq \delta\|Dw\|^q_{L^q(U)} - \beta|U|$ for some constant $\delta > 0$, which in turn guarantees that $J[w] \longrightarrow \infty$ as $\|Dw\|_{L^q(U)} \longrightarrow \infty$. Moreover, $J(\cdot)$ is also weakly lower semicontinuous provided that L is bounded from below and convex in the variable p. Hence, we can use a similar approach as before to get the following result.

Theorem 2.23. *Suppose L is convex in the variable p and satisfies* (2.45)*, and also suppose the admissible set $\mathcal{A}$ is non-empty. Then there exists at least one function $u \in \mathcal{A}$ satisfying*

$$J[u] = \min_{w\in\mathcal{A}} J[w].$$

Furthermore, this minimizer is unique if $L = L(p,x)$ does not depend on z and there exists $\theta > 0$ such that

$$\sum_{i,j=1}^{n} L_{p_i p_j}(p,x)\xi_i\xi_j \geq \theta|\xi|^2 \ \textit{ for } p,\xi \in \mathbb{R}^n,\ x \in U.$$

2.9.2 Existence of Minimizers Under Constraints

Using a similar variational approach, we establish the existence of weak solutions for the subcritical Lane-Emden equation.

Theorem 2.24. *Suppose that $U \subset \mathbb{R}^n$ is a bounded domain with smooth boundary ∂U and let $1 < p < \frac{n+2}{n-2}$. Then there exists a non-trivial weak solution $u \in H_0^1(U)$ of the semi-linear Dirichlet problem*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } U, \\ \quad u = 0 & \text{on } \partial U. \end{cases} \tag{2.46}$$

Remark 2.14. *We must be careful in setting up our variational procedure for this problem. For example, we can naively consider the functional*

$$J(u) = \frac{1}{2}\int_U |Du|^2\,dx - \frac{1}{p+1}\int_U |u|^{p+1}\,dx.$$

It is not to difficult to show that

$$\frac{d}{dt}J(u+tv)\Big|_{t=0} = \int_U Du \cdot Dv - |u|^{p-1}uv\,dx.$$

Therefore, a critical point of the functional J in $H := H_0^1(U)$ is a weak solution of (2.46). However, the functional J is not bounded from below in H. To see this, fix $u \in H$ and consider

$$J(tu) = \frac{t^2}{2}\int_U |Du|^2\,dx - \frac{t^{p+1}}{p+1}\int_U |u|^{p+1}\,dx.$$

Since $p+1 > 2$, we see that $J(tu) \longrightarrow -\infty$ as $t \longrightarrow \infty$. To get around this problem, we choose a different functional with constraints.

Proof. Set

$$I(u) = \frac{1}{2}\int_U |Du|^2\,dx$$

under the constraint

$$M := \{u \in H : G(u) := \int_U |u|^{p+1}\,dx = 1\}.$$

We seek minimizers of I in M. Let $\{u_k\} \subset M$ be a minimizing sequence. It follows that $\int_U |Du_k|^2\,dx$ is bounded so that $\{u_k\}$ is bounded in H. By the weak-$*$ compactness of bounded sets in the reflexive Hilbert space H, u_k converges weakly to some u_0 in H. Thus, the weak lower semi-continuity of the functional I implies that

$$I(u_0) \le \liminf_{k\longrightarrow\infty} I(u_k) =: m. \tag{2.47}$$

Since $p+1 < \frac{2n}{n-2}$, the compact Sobolev embedding theorem implies that $H^1(U)$ is compactly embedded in $L^{p+1}(U)$. Therefore, u_k converges strongly to u_0 in $L^{p+1}(U)$, which implies $u_0 \in M$ since

$$1 = \int_U |u_k|^{p+1}\,dx \longrightarrow \int_U |u_0|^{p+1}\,dx \text{ as } k \longrightarrow \infty.$$

Thus, $I(u_0) \geq m$. Combining this with (2.47) yields $I(u_0) = m$. This proves the existence of a minimizer u_0 of I in M. It remains to show that u_0, multiplied by a suitable constant if necessary, is a non-trivial weak solution of (2.46). This entails identifying the corresponding Euler–Lagrange equation for this minimizer under the constraint, which is provided by the following theorem whose proof is given on page 60 in [6].

Theorem 2.25 (Lagrange Multiplier)**.** *Let u be a minimizer of I in M, i.e.,*

$$I(u) = \min_{v \in M} I(v).$$

Then there exists a real number λ such that

$$I'(u) = \lambda G'(u)$$

or

$$\langle I'(u), v\rangle = \lambda \langle G'(u), v\rangle \text{ for all } v \in H.$$

We are now ready to show the minimizer u_0 is a weak solution of (2.46) after a suitable dilation. The minimizer u_0 of I under the constraint $G(u) = 1$ satisfies the Euler–Lagrange equation

$$\langle I'(u_0), v\rangle = \lambda \langle G'(u_0), v\rangle \text{ for all } v \in H;$$

that is,

$$\int_U Du_0 \cdot Dv\,dx = \lambda \int_U |u_0|^{p-1} u_0 v\,dx \text{ for all } v \in H.$$

From this, we can choose $v = u_0$ so that

$$\lambda = \frac{\displaystyle\int_U |Du_0|^2\,dx}{\displaystyle\int_U |u_0|^{p+1}\,dx},$$

and thus $\lambda > 0$. Then we can set $\tilde{u} = au_0$ where $\lambda / a^{p-1} = 1$ since $p > 1$. Hence

$$\int_U D\tilde{u} \cdot Dv\,dx = \int_U |\tilde{u}|^{p-1} \tilde{u} v\,dx,$$

so $\tilde{u} \in H$ is a weak solution of (2.46). □

2.10 Calculus of Variations II: Critical Points and Mountain Pass Theorems

In the previous examples, we obtained minimizers to a given functional, which turn out to be weak solutions to a corresponding PDE. More generally, we also showed that the critical points of the functional are also weak solutions. In applications, however, there are often situations were the functionals considered are not bounded above or below and so the existence of local minima is no longer guaranteed. Fortunately, critical points may still exist in the form of so-called saddle points. this section, we use the celebrated Mountain Pass theorem of Ambrosetti and Rabinowitz (see [2]) and its variants to find such critical points. In order to state and prove the Mountain Pass theorem, we first need to introduce some definitions and an important deformation theorem.

2.10.1 Deformations and Mountain Pass Theorem

Hereafter, H denotes a Hilbert space with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$ and $I : H \longrightarrow \mathbb{R}$ is a nonlinear functional on H.

Definition 2.8. *We say I is differentiable at $u \in H$ if there exists $v \in H$ such that*

$$I[w] = I[u] + (v, w-u) + o(\|w-u\|) \text{ for } w \in H. \tag{2.48}$$

The element v, if it exists, is unique and we write $I'[u] = v$.

Definition 2.9. *We say I belongs to $C^1(H;\mathbb{R})$ if $I'[u]$ exists for each $u \in H$ and the mapping $I' : H \longrightarrow H$ is continuous.*

Remark 2.15. *(a) The results we develop in this section hold assuming $I \in C^1(H:\mathbb{R})$, but for simplicity, we shall assume additionally that $I' : H \longrightarrow H$ is Lipschitz continuous on bounded subsets of H. Moreover, we denote by $\mathcal{C}$ the collection of all such I satisfying these conditions.*

(b) If $c \in \mathbb{R}$, we set

$$A_c := \{u \in H \mid I[u] \le c\} \text{ and } K_c := \{u \in H \mid I[u] = c,\ I'[u] = 0\}.$$

Definition 2.10. *We say $u \in H$ is a critical point if $I'[u] = 0$. The real number c is a critical value if $K_c \neq \emptyset$.*

In general, H is taken to be an infinite dimensional Hilbert space, thus we need to impose some sort of compactness condition.

Definition 2.11 (Palais-Smale)**.** *A functional $I \in C^1(H;\mathbb{R})$ satisfies the Palais-Smale compactness condition, or (PS) condition for short, if each sequence $\{u_k\}_{k=1}^\infty \subset H$ such that* *(a)* $\{I[u_k]\}_{k=1}^{\infty}$ *is bounded,*

(b) $I'[u_k] \longrightarrow 0$ *in* H*,*

is precompact in H*. We will sometimes call such a sequence satisfying (a)-(b) a Palais-Smale or* (PS) *sequence.*

Similarly, if we replace (a) above with (a') $I[u_k] \longrightarrow c$ *for some* $c \in \mathbb{R}$*, then we say* I *satisfies the* $(PS)_c$ *condition, and we call such a sequence satisfying* $(a') - (b)$ *a* $(PS)_c$ *sequence.*

The following theorem states that if c is not a critical value, we can deform the set $A_{c+\epsilon}$ into $A_{c-\epsilon}$ for some $\epsilon > 0$. The principle idea lies around solving an ODE in H.

Theorem 2.26 (Deformation)**.** *Assume* $I \in \mathcal{C}$ *satisfies the Palais-Smale condition and suppose that* $K_c = \emptyset$*. Then for each sufficiently small* $\epsilon > 0$*, there exists a constant* $\delta \in (0, \epsilon)$ *and a deformation function*

$$\eta \in C([0,1] \times H; H)$$

such that the mappings

$$\eta_t(u) = \eta(t,u) \ \textit{for} \ t \in [0,1],\, u \in H$$

satisfy

(i) $\eta_0(u) = u$ *for* $u \in H$*,*

(ii) $\eta_1(u) = u$ *for* $u \notin I^{-1}([c-\epsilon,\, c+\epsilon])$*,*

(iii) $I[\eta_t(u)] \leq I[u]$ *for* $t \in [0,1]$, $u \in H$*,*

(iv) $\eta_1(A_{c+\delta}) \subset A_{c-\delta}$*.*

Proof. **Step 1:** We claim that there exist constants $\sigma, \epsilon \in (0,1)$ such that

$$\|I'[u]\| \geq \sigma \ \text{ for each } \ u \in A_{c+\epsilon} - A_{c-\epsilon}. \tag{2.49}$$

To see this, we proceed by contradiction. Assume (2.49) were false for all constant $\sigma, \epsilon > 0$. Then there would exist sequences $\sigma_k \to 0$ and $\epsilon_k \to 0$ and elements

$$u_k \in A_{c+\epsilon_k} - A_{c-\epsilon_k} \ \text{ with } \ \|I'[u_k]\| \leq \sigma_k. \tag{2.50}$$

According to the Palais-Smale condition, there is a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ and an element $u \in H$ such that $u_k \to u$ in H. Since $I \in C^1(H;\mathbb{R})$, (2.50) implies that $I[u] = c$ and $I'[u] = 0$. Hence, $K_c \neq \emptyset$ and we arrive at a contradiction.

Step 2: Now fix δ such that

$$\delta \in (0,\epsilon) \ \text{ and } \ \delta \in (0, \sigma^2/2). \tag{2.51}$$

Denote

$$A := \{u \in H \mid I[u] \le c - \epsilon \text{ or } I[u] \ge c + \epsilon\},$$
$$B := \{u \in H \mid c - \delta \le I[u] \le c + \delta\}.$$

Since I' is bounded on bounded sets, we verify that the mapping $u \mapsto dist(u, A) + dist(u, B)$ is bounded from below by a positive constant on each bounded subset of H. Therefore, the function,

$$g(u) = \frac{dist(u, A)}{dist(u, A) + dist(u, B)}, \ \ (u \in H),$$

is Lipschitz continuous on bounded sets and satisfies

$$0 \le g \le 1, \ g = 0 \text{ on } A, \ g = 1 \text{ on } B.$$

Now set

$$h(t) = \begin{cases} 1, & \text{if } t \in [0, 1], \\ 1/t, & \text{if } t \ge 1, \end{cases} \tag{2.52}$$

and define the bounded operator $V : H \to H$ by

$$V(u) = -g(u)h(\|I'[u]\|)I'[u] \ (u \in H). \tag{2.53}$$

Consider, for each $u \in H$, the abstract ordinary differential equation

$$\begin{cases} \frac{d}{dt}\eta(t) = V(\eta(t)) \quad t > 0, \\ \eta(0) = u. \end{cases} \tag{2.54}$$

Indeed, there exists a unique global solution $\eta = \eta(t, u) = \eta_t(u)$ for $t \ge 0$, since V is bounded and Lipschitz continuous on bounded sets. Moreover, if we restrict our attention to the smaller interval $t \in [0, 1]$, it is easy to see that $\eta \in C([0, 1] \times H; H)$ and satisfies assertions (i) and (ii).

Step 3: It remains to verify assertions (iii) - (iv).

There holds

$$\frac{d}{dt}I[\eta_t(u)] = I'[\eta_t(u)] \cdot \frac{d}{dt}\eta_t(u) = I'[\eta_t(u)] \cdot V(\eta_t(u)) = -g(\eta_t(u))h(\|I'[\eta_t(u)]\|)\|I'[\eta_t(u)]\|^2. \tag{2.55}$$

In particular,

$$\frac{d}{dt}I[\eta_t(u)] \le 0 \text{ for } u \in H, \ t \in [0, 1],$$

and this verifies assertion (iii).

Now fix any point $u \in A_{c+\delta}$. We claim that $\eta_1(u) \in A_{c-\delta}$, i.e., assertion (iv) holds. To see this, if $\eta_t(u) \notin B$ for some $t \in [0, 1]$, we are done. So, instead, assume that $\eta_t(u) \in B$ for all $t \in [0, 1]$. Then $g(\eta_t(u)) = 1$ for all $t \in [0, 1]$. Hence, identity (2.55) implies that

$$\frac{d}{dt}I[\eta_t(u)] = -h(\|I'[\eta_t(u)]\|)\|I'[\eta_t(u)]\|^2. \tag{2.56}$$

If $\|I'[\eta_t(u)]\| \geq 1$, then (2.49) and (2.52) imply that

$$\frac{d}{dt} I[\eta_t(u)] = -\|I'[\eta_t(u)]\|^2 \leq -\sigma^2.$$

Likewise, if $\|I'[\eta_t(u)]\| \leq 1$, then (2.49) and (2.52) also imply that

$$\frac{d}{dt} I[\eta_t(u)] \leq -\sigma^2.$$

These two inequalities, when combined with (2.51) and (2.56), imply

$$I[\eta_1(u)] \leq I[u] - \sigma^2 \leq c + \delta - \sigma^2 \leq c - \delta.$$

This verifies the claim that $\eta_1(u) \in A_{c-\delta}$ and this completes the proof. □

With the help of the Deformation Theorem, we shall now prove the celebrated Mountain Pass Theorem, which guarantees the existence of a critical point.

Theorem 2.27 (Mountain Pass)**.** *Assume $I \in \mathcal{C}$ satisfies the Palais-Smale condition. Suppose, in addition, that*

(i) $I[0] = 0$,

(ii) there exist constants $a, r > 0$ such that

$$I[u] \geq a \;\; \textit{if} \;\; \|u\| = r,$$

(iii) there exists an element $v \in H$ with

$$\|v\| > r,\; I[v] \leq 0.$$

Then

$$c = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I[g(t)],$$

where

$$\Gamma := \{g \in C([0,1]; H) \,|\, g(0) = 0,\; g(1) = v\},$$

is a critical value of I.

Proof. Indeed, it is clear that $c \geq a$. Now assume that c is not a critical value of I so that $K_c = \emptyset$. Choose a suitably small $\epsilon \in (0, a/2)$. According to the deformation theorem, there exists a constant $\delta \in (0, \epsilon)$ and a homeomorphism $\eta : H \to H$ with

$$\eta(A_{c+\delta}) \subset A_{c-\delta}$$

and

$$\eta(u) = u \;\text{ if }\; u \notin I^{-1}[c - \epsilon, c + \epsilon]. \tag{2.57}$$

Now select $g \in \Gamma$ such that

$$\max_{0 \le t \le 1} I[g(t)] \le c + \delta. \tag{2.58}$$

Then the composition

$$\hat{g} = \eta \circ g$$

is also in Γ, since $\eta(g(0)) = \eta(0) = 0$ and $\eta(g(1)) = \eta(v) = v$ as indicated in (2.57). But then (2.58) implies that

$$\max_{0 \le t \le 1} I[\hat{g}(t)] \le c - \delta,$$

and so

$$c = \inf_{g \in \Gamma} \max_{0 \le t \le 1} I[g(t)] \le c - \delta,$$

which is a contradiction. □

Remark 2.16. *The rough picture of the result above is to view $0 \in H$ (basin) surrounded by a mountain range (the sphere $\partial B_r(0) \subset H$). We are assuming the elevation level of $I : H \longrightarrow \mathbb{R}$ restricted to the mountain range is positive and bounded away from zero. Assuming there is a location (the element v) outside the basin and mountain ridge whose elevation is even lower, Theorem 2.27 reveals the shortest path (mountain pass) across the mountains starting from 0 to v produces the desired critical value and critical point.*

2.10.2 Application of the Mountain Pass Theorem

We will prove the existence of at least one non-trivial weak solution to a general semilinear boundary value problem in which the Lane-Emden equation is a special case. Namely, consider the boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2.59}$$

We assume f is smooth, and for some $1 < p < \frac{n+2}{n-2}$, there holds for some positive constant C,

$$|f(z)| \le C(1 + |z|^p),\ |f'(z)| \le C(1 + |z|^{p-1}) \ \text{ for } \ z \in \mathbb{R}. \tag{2.60}$$

If we denote

$$F(z) = \int_0^z f(s)\, ds \ \text{ and } \ z \in \mathbb{R},$$

we also assume that

$$0 \le F(z) \le \gamma f(z) z \ \text{ for some constant } \ \gamma < 1/2, \tag{2.61}$$

and for constants $0 < a \le A$,

$$a|z|^{p+1} \le |F(z)| \le A|z|^{p+1} \ \text{ for } \ z \in \mathbb{R}. \tag{2.62}$$

Remark 2.17. *(a) Indeed,* (2.62) *implies that* $f(0) = 0$ *and so* $u \equiv 0$ *is a trivial solution of* (2.59).

(b) It is easy to check that $f(u) = |u|^{p-1}u$ *satisfies the above conditions.*

Theorem 2.28. *The boundary value problem* (2.59) *has at least one non-trivial weak solution.*

The basic idea of the proof is to consider the functional

$$I[u] := \int_U \frac{1}{2}|Du|^2 - F(u)\,dx \text{ for } u \in H, \tag{2.63}$$

where $H = H_0^1(U)$ with the induced norm coming from the inner product $(u, v) = \int_U Du \cdot Dv\,dx$, then show that the Mountain Pass Theorem applies. Therefore, the existence of a non-trivial critical point of I implies the existence of a non-trivial weak solution of the boundary value problem. To best illustrate the main ingredients of the proof, we introduce the following lemmas.

Lemma 2.5. *There hold* $I[0] = 0$ *and* I *belongs to the class* $\mathcal{C}$*.*

Proof. It is obvious that $I[0] = 0$. It remains to show that $I \in \mathcal{C}$. Consider the splitting

$$I[u] = \frac{1}{2}\|u\|^2 - \int_U F(u)\,dx := I_1[u] + I_2[u].$$

Indeed, for $u, w \in H$,

$$I_1[w] = \frac{1}{2}\|w\|^2 = \frac{1}{2}\|u+w-u\|^2 = \frac{1}{2}\|u\|^2+(u, w-u)+\frac{1}{2}\|w-u\|^2 = I_1[u]+(u, w-u)+o(\|w-u\|).$$

Therefore, I_1 is differentiable at u with $I_1'[u] = u$, and thus $I_1 \in \mathcal{C}$. Now we show $I_2 \in \mathcal{C}$. First we make some preliminary observations. Recall that the Lax-Milgram theorem states that for each element $v^* \in H^{-1}(U)$, the boundary value problem,

$$\begin{cases} -\Delta v = v^* & \text{in } U, \\ \quad v = 0 & \text{on } \partial U. \end{cases} \tag{2.64}$$

has a unique solution $v \in H_0^1(U)$. Write $v = Kv^*$ so that

$$K : H^{-1}(U) \to H_0^1(U) \tag{2.65}$$

is an isometry. In particular, recall that if $w \in L^{\frac{2n}{n+2}}(U)$, then the linear functional w^* defined by

$$(w^*,\, u) := \int_U wu\,dx \text{ for } u \in H_0^1(U)$$

belongs to $H^{-1}(U)$. Here we shall abuse conventional notation and say that w belongs to $H^{-1}(U)$. In addition, the subcritical condition implies that $p(\frac{2n}{n+2}) < \frac{2n}{n-2}$ and so $f(u)$ belongs to $L^{\frac{2n}{n+2}}(U) \subset H^{-1}(U)$ provided that $u \in H_0^1(U)$. The crucial step here is that

$$I_2'[u] = K[f(u)]. \tag{2.66}$$

To see this, notice that

$$F(a+b) = F(a) + f(a)b + \int_0^1 (1-s)f'(a+sb)\,dsb^2$$

and thus for each $w \in H_0^1(U)$,

$$\begin{aligned} I_2[w] &= \int_U F(w)\,dx = \int_U F(u+w-u)\,dx = \int_U F(u) + f(u)(w-u)\,dx + R \\ &= I_2(u) + \int_U DK[f(u)] \cdot D(w-u)\,dx + R, \end{aligned}$$

where the remainder term R, according to (2.60), satisfies

$$\begin{aligned} |R| &\le C \int_U (1 + |u|^{p-1} + |w-u|^{p-1})|w-u|^2\,dx \\ &\le C_1\Big(\int_U |w-u|^2 + |w-u|^{p+1}\,dx\Big) + C_2\Big(\int_U |u|^{p+1}\,dx\Big)^{\frac{p-1}{p+1}}\Big(\int_U |w-u|^{p+1}\,dx\Big)^{\frac{2}{p+1}}. \end{aligned}$$

Hence, since $p+1 < \frac{2n}{n-2}$, Sobolev embedding implies that $R = o(\|w-u\|)$. Therefore,

$$I_2[w] = I_2[w] + (K[f(u)],\, w-u) + o(\|w-u\|).$$

Lastly, if $u, v \in B_L(0) \subset H_0^1(U)$, then

$$\|I_2'[u] - I_2'[v]\| = \|K[f(u)] - K[f(v)]\|_{H_0^1(U)} = \|f(u) - f(v)\|_{H^{-1}(U)} \le \|f(u) - f(v)\|_{L^{\frac{2n}{n+2}}}.$$

Furthermore, (2.60) and Hölder's inequality imply

$$\begin{aligned} \|f(u) - f(v)\|_{L^{\frac{2n}{n+2}}(U)} &\le C\Big(\int_U ((1 + |u|^{p-1} + |v|^{p-1})|u-v|)^{\frac{2n}{n+2}}\,dx\Big)^{\frac{n+2}{2n}} \\ &\le C\Big(\int_U ((1 + |u|^{p-1} + |v|^{p-1})|u-v|)^{\frac{2n}{n+2}\frac{n+2}{4}}\,dx\Big)^{\frac{2}{n}} \|u-v\|_{L^{\frac{2n}{n-2}}(U)} \\ &\le C(L)\|u-v\|_{L^{\frac{2n}{n-2}}(U)} \\ &\le C(L)\|u-v\|. \end{aligned}$$

This shows that $I_2' : H_0^1(U) \to H_0^1(U)$ is Lipschitz continuous on bounded sets and thus, $I_2 \in \mathcal{C}$. This completes the proof of the lemma. □

Lemma 2.6. *The functional $I \in \mathcal{C}$ satisfies the Palais-Smale condition.*

Proof. Suppose the sequence $\{u_k\}_{k=1}^{\infty}$ in $H_0^1(U)$ satisfies

$$(i)\ \{I[u_k]\}_{k=1}^{\infty} \text{ is bounded, and } (ii)\ I'[u_k] \to 0 \text{ in } H_0^1(U). \tag{2.67}$$

Obviously, we have that

$$u_k - K(f(u_k)) \to 0 \text{ in } H_0^1(U). \tag{2.68}$$

Thus, for each $\epsilon > 0$, we have

$$|(I'[u_k],\, v)| = \left| \int_U Du_k \cdot Dv - f(u_k)v\, dx \right| \leq \epsilon \|v\| \text{ for } v \in H_0^1(U)$$

for sufficiently large k. Namely, if we take $v = u_k$ and set $\epsilon = 1$, then we get

$$\left| \int_U |Du_k|^2 - f(u_k)u_k\, dx \right| \leq \|u_k\|$$

for sufficiently large k. From (2.67), we have that

$$\Big(\frac{1}{2}\|u_k\|^2 - \int_U F(u_k)\, dx \Big) \leq C < \infty$$

for all k. Hence, we deduce from above and (2.61) that

$$\|u_k\|^2 \leq C + 2\int_U F(u_k)\, dx \leq C + 2\gamma(\|u_k\|^2 + \|u_k\|).$$

As $2\gamma < 1$, we can absorb the last two terms on the right-hand side by the left-hand side to get that $\{u_k\}_{k=1}^{\infty}$ is bounded in $H_0^1(U)$. We can then extract a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$, that converges weakly to $u \in H_0^1(U)$. Hence, $u_{k_j} \longrightarrow u$ in $L^{p+1}(U)$ since $p + 1 < \frac{2n}{n-2}$ by the compact Sobolev embedding. But then $f(u_{k_j}) \longrightarrow f(u)$ in $H^{-1}(U)$ and so $K[f(u_{k_j})] \longrightarrow K[f(u)]$ in $H_0^1(U)$. Consequently, from (2.68), we arrive at the desired conclusion that

$$u_{k_j} \to u \text{ in } H_0^1(U). \tag{2.69}$$

□

Lemma 2.7. *There hold the following statements.*

(a) There exist constants $r, a > 0$ such that

$$I[u] \geq a \ \textit{if}\ \|u\| = r.$$

(b) There exists an element $v \in H_0^1(U)$ with

$$\|v\| > r \ \textit{and}\ I[v] \leq 0.$$

Proof. (i) Suppose that $u \in H_0^1(U)$ with $\|u\| = r$ for some $r > 0$ to be determined below. Then

$$I[u] = I_1[u] - I_2[u] = \frac{r^2}{2} - I_2[u].$$

By (2.62) and Sobolev embedding, as $p + 1 < \frac{2n}{n-2}$, we obtain that

$$|I_2[u]| \le C \int_U |u|^{p+1}\,dx \le C\Big(\int_U |u|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{(p+1)(n-2)}{2n}} \le C\|u\|^{p+1} \le Cr^{p+1}.$$

Hence,

$$I[u] \ge \frac{r^2}{2} - Cr^{p+1} \ge \frac{r^2}{4} = a > 0,$$

provided that $r > 0$ is chosen small enough, since $p + 1 > 2$.

(ii) Fix some non-trivial element $u \in H_0^1(U)$ and write $v = tu$ for $t > 0$ to be determined below. Then, using (2.62), we get

$$I[v] = I_1[tu] - I_2[tu] = t^2 I_1[u] - \int_U F(tu)\,dx \le t^2 I_1[u] - at^{p+1}\int_U |u|^{p+1}\,dx < 0$$

for $t > 0$ large enough. □

Proof of Theorem 2.28. Indeed, Lemmas 2.5–2.7 verify all the hypotheses in the Mountain Pass theorem. Hence, the Mountain Pass theorem implies there exists a non-trivial function $u \in H_0^1(U)$ with

$$I'[u] = u - K[f(u)] = 0.$$

In particular, for each $v \in H_0^1(U)$, there holds

$$\int_U Du \cdot Dv\,dx = \int_U f(u)v\,dx,$$

and so u is a non-trivial weak solution of the boundary value problem (2.59). □

2.10.3 Linking theorems and their applications

The Mountain Pass Theorem turns out to be a special case of an even more general family of min-max techniques called linking theorems. Here we shall assume $I \in C^1(H;\mathbb{R})$ and let $\mathcal{C}$ be a non-empty collection of subsets $A \subset H$ such that (1) $c := \inf_{A\in\mathcal{C}} \sup_{u\in A} I(u)$ is finite, and (2) if η denotes a deformation formation (obtained from a descent flow of I), then $\eta(A) \in \mathcal{C}$ for all $A \in \mathcal{C}$. Then it follows that c is a critical value of I provided that I satisfies the $(PS)_c$ condition.

Below, we let N be a manifold with boundary ∂N, C is a non-empty subset of H and $\Gamma = \{h \in C(N;E)\,|\,h \equiv id \text{ on } \partial N\}$. We first define the meaning of 'linking' sets.

Definition 2.12. *We say that ∂N and C link if*

$$C \cap h(N) \neq \emptyset \quad \textit{for all } h \in \Gamma.$$

Remark 2.18. *As a precursor to applications below, consider the standard example: let $H = V \oplus W$ be a Hilbert space, where V, W are orthogonal closed subspaces and $dim(V) = k < \infty$. Given $e \in W$ and $R > 0$, we consider the $k+1$ dimensional manifold with boundary,*

$$N = \{z = v + se \,|\, v \in V, \|v\| \leq R, 0 \leq s \leq 1\}.$$

Letting C be the sphere $B_r(0) \subset W$, i.e.,

$$C = \{w \in W \,|\, \|w\| = r\},$$

then ∂N and C link. This can be verified using a topological degree argument but the details are left to the reader.

Definition 2.13. *Define*

$$c := \inf_{h \in \Gamma} \sup_{u \in N} I(h(u)). \tag{2.70}$$

This value c is called the linking level of I (with respect to N and C).

We are now prepared to introduce the linking theorems and their connection to critical points of I. Again, let $I \in C^1(H; \mathbb{R})$ and let N and C be subsets of H such that ∂N and C link. For the rest of this subsection, we assume the following.

(L1) I is bounded from below on C i.e.,

$$\rho := \inf_{u \in C} I(u) > -\infty,$$

(L2) $\rho > \beta := \sup_{u \in \partial N} I(u)$.

If ∂N and C link so that $C \cap h(N) \neq \emptyset$ for all $h \in \Gamma$, then we easily see that

$$\sup_{u \in N} I(h(u)) \geq \inf_{u \in C \cap h(N)} I(u) \geq \inf_{u \in C} I(u) = \rho.$$

That is,

$$c \geq \rho.$$

Theorem 2.29. *Let ∂N and C link and suppose $I \in C^1(H; \mathbb{R})$ satisfy (L1)−(L2). Moreover, assume that I satisfies the $(PS)_c$ condition with linking level c as defined in* (2.70)*. Then c is a critical value of I such that $c \geq \rho$.*

Proof. The proof is similar to that of Theorem 2.27. Proceeding by contradiction, we assume that the set K_c of critical points at level c is empty. As in the proof of the deformation lemma, there exists a deformation function η and a suitably small $\delta > 0$ satisfying

(i) $\eta(A_{c+\delta}) \subset A_{c-\delta}$,

(ii) $\eta(u) = u$ for all $u \in A_\beta$.

From the latter property, we can show $\eta \circ h \in \Gamma$ for each $h \in \Gamma$. In particular, if $u \in \partial N$ and $I(u) \leq \beta$, then the second property implies $\eta \circ h(u) = u$, i.e., the composition is the identity on the boundary. By definition, we can pick $h \in \Gamma$ such that

$$\sup_{u \in N} I(h(u)) \leq c + \delta.$$

From (i) we deduce that

$$\sup_{u \in N} I(\eta \circ h(u)) \leq c - \delta,$$

but this contradictions with the definition of the linking level c, since $\eta \circ h \in \Gamma$. □

Remark 2.19. *The Mountain Pass Theorem (see Theorem 2.27) is a special case of the Linking Theorem of Theorem 2.29. To see this, just take $v \in H$, $N = [0, v] := \{tv \mid 0 \leq t \leq 1\}$ so that $\partial N = \{0, v\}$, $C = \partial B_r(0) = \{u \in H \mid \|u\| = r\}$ and $\Gamma = \{h \in C([0, v]; H)\, h(0) = 0, h(v) = v\}$. Indeed, the sets ∂N and C link provided that $\|v\| > r$.*

Theorem 2.30. *Let $H = V \oplus W$ be a Hilbert space, where V, W are closed subspaces and $dim(V) = k < \infty$. Suppose that $I \in C^1(H; \mathbb{R})$ satisfies the following.*

(L1') There exist $\rho, r > 0$ such that

$$J(w) \geq \rho \quad \text{for all } w \in W \quad \text{with } \|w\| = r,$$

(L2') there exist $R > 0$ and $w \in W$ with $\|w\| > r$ such that, letting

$$N = \{u = v + tw \mid v \in V, \|v\| \leq R, 0 \leq t \leq 1\},$$

there holds

$$I(u) < 0 \quad \text{for all } u \in \partial N.$$

Further assume I satisfies the $(PS)_c$ condition. Then I has a non-trivial critical point z at level $c > 0$.

We now apply the previous results to establish some existence result to a familiar boundary-value problem. To illustrate how the Linking Theorems enhance the Mountain Pass Theorem, we recall the following result, which is a special case of Theorem 2.28.

Theorem 2.31. *Let $U \subset \mathbb{R}^n$ be a C^1, open and bounded domain, and suppose $1 < p < 2^* - 1 := (n+2)/(n-2)$ and $\lambda < \lambda_1$, where λ_1 is the principle eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Then the boundary-value problem,*

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-1} u & \text{in } U, \\ \quad u = 0 & \text{on } \partial U, \end{cases} \tag{2.71}$$

admits a positive weak solution.

The above result is a consequence of the Mountain Pass Theorem. However, with the help of Theorem 2.30, we improve the result by removing the restriction that λ lies below the first Dirichlet eigenvalue for the Laplace operator.

Theorem 2.32. *Let $U \subset \mathbb{R}^n$ be a C^1, open and bounded domain, and suppose $1 < p < 2^* - 1$ and $\lambda \in \mathbb{R}$. Then the boundary-value problem* (2.71) *admits a non-trivial weak solution.*

Proof. Let $H := H_0^1(U)$ and set

$$I(u) = \frac{1}{2}\int_U |\nabla u|^2\,dx - \frac{\lambda}{2}\int_U u^2\,dx - \frac{1}{p+1}\int_U |u|^{p+1}\,dx$$

where λ is any real number. Recall, we let $\lambda_1 < \lambda_2 \le \lambda_3 \le \dots$ and $\varphi_1, \varphi_2, \varphi_3, \dots$ represent the Dirichlet eigenvalues and associated eigenfunctions of $-\Delta$ in H. Therefore, there is a natural k such that $\lambda_k \le \lambda < \lambda_{k+1}$.

Our goal is to show for each such $\lambda_k \le \lambda < \lambda_{k+1}$, we can apply Theorem 2.30 with

$$V = span\{\varphi_1, \varphi_2, \dots, \varphi_k\} \quad \text{and} \quad W = V^{\perp},$$

the orthogonal complement of V in $L^2(U)$. We adopt the same choices for the subsets N and C in Remark 2.18.

We verify $(L1')$ holds. Indeed, if $w = \sum_{i=k+1}^{\infty} c_i\varphi_i \in W$ and $\|w\|_H = o(1)$, then

$$I(w) = \frac{1}{2}\sum_{i=k+1}^{\infty} c_i^2\Big(1 - \frac{\lambda}{\lambda_i}\Big) + o(\|w\|_H^2) \ge \frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_{k+1}}\Big) + o(\|w\|_H^2).$$

This verifies the first property. It remains to verify $(L2')$ holds. First, we take any finite dimensional subspace V' of H. For $v' \in V'$, $\|v'\|_H = 1$, there holds

$$I(Rv') = \frac{1}{2}R^2 - \frac{1}{2}\lambda^2 R^2\|v'\|_{L^2(U)}^2 - \frac{1}{p+1}R^{p+1}\|v'\|_{L^{p+1}(U)}^{p+1}. \tag{2.72}$$

Since $p > 1$ and V' has finite dimension, from (2.72) we can find $R > 0$ such that $I(Rv') < 0$ for all such $v' \in V'$ with $\|v'\|_H = 1$. Particularly, we can find $R > r > 0$ and $w \in W$ with $\|w\|_H = R$ such that $I(v + tw) < 0$ if $\|v + tw\| \ge R$. We can also check that on the three sides of ∂N given by $\{v + tw \mid \|v\|_H = R\} \cap \{v + Rw\}$, we have $I(u) \le 0$.

However, for $v = \sum_{i=1}^{k} c_i\varphi_i \in V$, we also have that

$$\|v\|_{L^2(U)}^2 = \sum_{i=1}^{k} \lambda_i^{-1}c_i^2 > \lambda_k^{-1}\|v\|_H^2$$

and thus

$$I(v) \le \frac{1}{2}\|v\|_H^2 - \frac{\lambda}{2}\|v\|_{L^2(U)}^2 \le \frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_k}\Big)\|v\|_H^2 < 0.$$

This verifies the second property holds. The verification that I satisfies the $(PS)_c$ condition is similar to what was done in the proof of Theorem 2.28.

Finally, we apply Theorem 2.30 to obtain a non-trivial critical point of I in H, which provides the desired non-trivial solution of (2.71). $\square$

2.11 Calculus of Variations III: Concentration Compactness

In our variational approach for establishing the existence of solutions to semilinear equations, we exploited the compact Sobolev embedding due to the subcritical exponent p. In the critical setting, however, this compactness property fails. Fortunately, we can apply the principle of concentration compactness to recover the compactness of the minimizing sequence in the strong topology of $H_0^1(U)$. In Chapter 6, we look at this precise problem of concentration phenomena and how it relates to the breakdown of the compactness of critical Sobolev embeddings. More precisely, there we examine finding extremal functions to a constrained energy functional for a critical Sobolev inequality. Then we use the concentration compactness principle to recover strong convergence of a minimizing sequence to obtain a minimizer for the functional.

For now, we illustrate how to apply the concentration compactness principle to establish an existence result for a model elliptic problem. Namely, we consider the stationary Schrödinger equation

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-1}u \text{ in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{2.73}$$

where $n \geq 3$, $\lambda < 0$ and $p > 1$.

We first begin with some background and motivation. The well-known nonlinear Schrödinger (NLS) equation is given by

$$\begin{cases} i\partial_t v + \Delta v = \pm|v|^{p-1}v & (x,t) \text{ in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = \varphi(x) & \text{in } H_0^1(\mathbb{R}^n), \end{cases} \tag{2.74}$$

where solutions are understood in the usual weak or distributional sense. We say the nonlinearity in equation (2.74) is focusing or defocusing, respectively, if the right-hand side is $-|v|^{p-1}v$ or $+|v|^{p-1}v$, but we shall only concern ourselves with the focusing case. In either case, however, a key feature of the NLS equation is that mass and energy are conserved quantities, i.e., $M(v(t)) = M(v(0))$ and $E(v(t)) = E(v(0))$ where

$$M(v(t)) = \int_{\mathbb{R}^n} |v(x,t)|^2\,dx$$

and

$$E(v(t)) = \int_{\mathbb{R}^n} |Dv(x,t)|^2\,dx \pm \frac{1}{p+1}\int_{\mathbb{R}^n} |v(x,t)|^{p+1}\,dx.$$

In the focusing case, we may search for solitary wave solutions of the form $v(x,t) = u(x)e^{-i\lambda t}$ where u is some function in $H^1(\mathbb{R}^n)$ and $\lambda < 0$. Then, it is simple to see that u satisfies

$$-\Delta u = \lambda u + |u|^{p-1}u \text{ in } \mathbb{R}^n. \tag{2.75}$$

Indeed, there does exist solutions to equation (2.75) whenever $1 < p < (n+2)/(n-2)$, and this can be established through various ODE or variational approaches. For the sake of illustration and to keep our presentation simple, we employ the concentration compactness principle of P. Lions to solve a closely related variational problem. Namely, for $n \geq 3$ and $1 < p < 1 + 4/n$, we look for minimizers of the energy functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} |Du|^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}\, dx$$

under the constraint $\|u\|_2^2 = \lambda$ for a fixed $\lambda > 0$. More precisely, we consider

$$I_\lambda = \inf\{E(u) \mid u \in H^1(\mathbb{R}^n),\ \|u\|_2^2 = \lambda\}. \tag{2.76}$$

We establish

Theorem 2.33. *Let $n \geq 3$ and let $p \in (1, 1+4/n)$ and $\lambda > 0$ be arbitrary. Then $I_\lambda > -\infty$ and for any minimizing sequence $\{u_k\}_{k=1}^\infty \subset H^1(\mathbb{R}^n)$ of (2.76), there exists a sequence of points $\{y_k\}_{k=1}^\infty \subset \mathbb{R}^n$ such that the translated sequence $\{u_k(\cdot + y_k)\}_{k=1}^\infty$ is relatively compact in $H^1(\mathbb{R}^n)$ and whose limit is a minimizer of $E(\cdot)$.*

Remark 2.20. *(a) If $1 < p < 1+4/n$ and for any $\lambda > 0$, we have that $I_\lambda < 0$ and is finite.*

(b) Unfortunately, if $p > 1+4/n$, then $I_\lambda = -\infty$ for any $\lambda > 0$, i.e., the energy functional is no longer bounded from below (and this illustrates the restriction on p). For $1+4/n \leq p < (n+2)/(n-2)$, we can circumvent this issue by minimizing a slightly different functional (see Theorem 2.41). For another similar problem that minimizes the Dirichlet integral over an appropriately chosen admissible set, we refer the reader to Section 6.3 in Chapter 6.

(c) These minimizers for $E(\cdot)$ are indeed weak solutions to equation (2.75) but for a completely different parameter λ. In particular, the parameter λ in the problem for I_λ ($\lambda > 0$) and equation (2.75) ($\lambda < 0$) are not the same and are opposite in sign.

We shall make use of the following concentration compactness principle which we state without proof [22, 23]. Essentially, this proposition asserts that there are three possibilities when given a bounded sequence in $H^1(\mathbb{R}^n)$. The usual strategy for our variational problem is to verify that the other two "bad" scenarios cannot happen and that only strong precompactness of the sequence must hold.

Proposition 2.2. *Let $\lambda > 0$ and suppose $\{u_k\}_{k=1}^\infty$ is a bounded sequence in $H^1(\mathbb{R}^n)$ such that $\|u_k\|_2^2 = \lambda$ for $k = 1, 2, 3, \ldots$. Then there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty$ satisfying one of the following three properties.*

(I) (Compactness) There exists $\{y_j\}_{j=1}^\infty \subset \mathbb{R}^n$ such that for any $\epsilon > 0$, there exists $R > 0$ such that

$$\int_{y_j + B_R(0)} |u_{k_j}|^2\, dx \geq \lambda - \epsilon \text{ for } j = 1, 2, 3, \ldots.$$

(II) (Vanishing) For all $R > 0$,

$$\lim_{j\to\infty} \sup_{y\in\mathbb{R}^n} \int_{y+B_R(0)} |u_{k_j}|^2\,dx = 0.$$

(III) (Dichotomy) There exist $\alpha \in (0,\lambda)$ and bounded sequences $\{u_j^1\}_{j=1}^\infty$ and $\{u_j^2\}_{j=1}^\infty$ in $H^1(\mathbb{R}^n)$ such that

(a) $\lim_{j\to\infty} \|u_{k_j} - (u_j^1 + u_j^2)\|_q \longrightarrow 0$ *for* $2 \le q < \dfrac{2n}{n-2}$*;*

(b) $\alpha = \lim_{j\to\infty} \|u_j^1\|^2_{L^2(\mathbb{R}^n)}$ *and* $\lambda - \alpha = \lim_{j\to\infty} \|u_j^2\|^2_{L^2(\mathbb{R}^n)}$*;*

(c) $\liminf_{j\to\infty} \int_{\mathbb{R}^n} \left\{ |Du_{k_j}|^2 - |Du_j^1|^2 - |Du_j^2|^2 \right\} dx \ge 0.$

Remark 2.21. *Roughly speaking, only three situations can occur for such a bounded sequence of functions. Either (I) the sequence of functions concentrate near the points $\{y_j\}$, (II) such concentration does not occur at any of the points $\{y_j\}$, or (III) some fraction $\lambda \in (0,1)$ concentrates near some points $\{y_j\}$ while the remaining part spreads away from these points.*

We shall also require the following intermediate result.

Lemma 2.8. *There holds $I_\lambda < I_\alpha + I_{\lambda-\alpha}$ for any $\lambda > 0$ and $\alpha \in (0,\lambda)$.*

Proof. Let $\alpha \in [\lambda/2, \lambda)$ and $\theta \in (1, \lambda/\alpha]$. Then

$$\begin{aligned} I_{\theta\alpha} &= \inf_{u\in H^1(\mathbb{R}^n), \|u\|^2_{L^2(\mathbb{R}^n)}=\theta\alpha} E(u) = \inf_{u\in H^1(\mathbb{R}^n), \|u\|^2_{L^2(\mathbb{R}^n)}=\alpha} E(\theta^{1/2}u) \\ &= \theta \inf_{u\in H^1(\mathbb{R}^n), \|u\|^2_{L^2(\mathbb{R}^n)}=\alpha} \left\{ E(u) - \frac{\theta^{(p-1)/2}}{p+1} \int_{\mathbb{R}^n} |u|^{p+1}\,dx \right\} \\ &< \theta I_\alpha, \end{aligned}$$

where we used the fact that $I_\alpha < 0$ as indicated in Remark 2.20. Hence,

$$I_\lambda < \frac{\lambda}{\alpha} I_\alpha = I_\alpha + \frac{\lambda-\alpha}{\alpha} I_\alpha \le I_\alpha + I_{\lambda-\alpha}$$

□

Proof of Theorem 2.33. We divide the proof into three main steps.

Step 1: Let $\{u_k\}_{k=1}^\infty$ be a minimizing sequence for the energy functional $E(\cdot)$. The boundedness of the minimizing sequence follows immediately since the sequences $\{E(u_k)\}_{k=1}^\infty$ and $\{\|Du_k\|_{L^2(\mathbb{R}^n)}\}_{k=1}^\infty$ are bounded. From the concentration compactness principle of Proposition 2.2, there are three possibilities that may occur. The goal is to show that (II) vanishing and (III) dichotomy do not happen and that (I) compactness occurs. Once this is verified, the result follows accordingly. Namely, as done in the preceding sections, we may exploit the structure of the energy functional $E(u)$ to show the strong precompactness of the minimizing sequence, i.e., the translated subsequence given in (I) converges to some $u \in H^1(\mathbb{R}^n)$ with $\|u\|_{L^2(\mathbb{R}^n)}^2 \leq \lambda$. As usual, the next step is to show that the limit point u is admissible, i.e., $\|u\|_{L^2(\mathbb{R}^n}^2 = \lambda$, but this is immediately deduced from case (I) of Proposition 2.2 and we are done. Thus, it only remains to show that (II) and (III) cannot happen.

Step 2: (III) dichotomy does not occur.

Assume the contrary. Let $\alpha_j > 0$ and $\beta_j > 0$ be such that $\|\alpha_j u_j^1\|_{L^2(\mathbb{R}^n)}^2 = \alpha$ and $\|\beta_j u_j^2\|_{L^2(\mathbb{R}^n)}^2 = \lambda - \alpha$. Then $\alpha_j, \beta_j \longrightarrow 1$ as $j \longrightarrow \infty$ and we have

$$E(u_{k_j}) \geq E(u_j^1) + E(u_j^2) + \gamma_j = E(\alpha_j u_j^1) + E(\beta_j u_j^2) + \gamma_j'$$

where $\gamma_j, \gamma_j' \longrightarrow 0$ as $j \longrightarrow \infty$. Hence,

$$I_\lambda = \lim_{j\to\infty} E(u_{k_j}) \geq \lim_{j\to\infty} [E(\alpha_j u_j^1) + E(\beta_j u_j^2)] \geq I_\alpha + I_{\lambda-\alpha},$$

but this contradicts with Lemma 2.8.

Step 3: (II) vanishing does not occur.

Assume otherwise. It suffices to show that if (II) holds, then $\|u_{k_j}\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \longrightarrow 0$ as $j \longrightarrow \infty$ because then $\liminf_{j\to\infty} E(u_{k_j}) \geq 0$ and we get a contradiction with the fact that $I_\lambda < 0$. Choose an arbitrary $R > 0$. For any $y \in \mathbb{R}^n$, the Sobolev inequality yields

$$\|u\|_{L^{p+1}(B_R(0))}^{p+1} \leq C(R)\Big(\|u\|_{L^2(y+B_R(0))}^{p+1} + \|u\|_{L^2(y+B_R(0))}^{p+1+n-\frac{n}{2}(p+1)} \|Du\|_{L^2(y+B_R(0))}^{\frac{n}{2}(p+1)-n}\Big).$$

Choose a sequence $\{z_r\}_{r=1}^\infty \subset \mathbb{R}^n$ such that

$$\mathbb{R}^n \subset \bigcup_{r=1}^\infty \{z_r + B_R(0)\}$$

and each point $x \in \mathbb{R}^n$ is contained in at most ℓ balls where ℓ is a fixed positive integer. Then, noting that $\epsilon_j := \sup_r \|u_{k_j}\|_{L^2(z_r+B_R(0))}^{p-1} \longrightarrow 0$ as $j \longrightarrow \infty$ and applying the preceding Sobolev inequality, we get

$$\begin{aligned}
\|u_{k_j}\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} &\leq \sum_{r=1}^\infty \|u_{k_j}\|_{L^{p+1}(z_r+B_R(0))}^{p+1} \\
&\leq C(R)\epsilon_j \sum_{r=1}^\infty \Big\{\|u_{k_j}\|_{L^2(z_r+B_R(0))}^2 + \|u_{k_j}\|_{L^2(z_r+B_R(0))}^{2+n-\frac{n}{2}(p+1)} \|Du_{k_j}\|_{L^2(z_r+B_R(0))}^{\frac{n}{2}(p+1)-n}\Big\} \\
&\leq C\epsilon_j \sum_{r=1}^\infty \int_{z_r+B_R(0)} [u_{k_j}^2 + |Du_{k_j}|^2]\, dx \leq C\ell\epsilon_j \|u_{k_j}\|_{L^1(\mathbb{R}^n)}^2 \longrightarrow 0
\end{aligned}$$

as $j \longrightarrow \infty$, where we used Jensen's inequality in the last line. This proves the claim.

Hence, u is a minimizer of $E(\cdot)$, i.e., $E(u) = I_\lambda$ as defined in problem (2.76). This completes the proof. □

2.12 Sharp Existence Results and A Priori Estimates

2.12.1 Sharp Existence Results

The Lane-Emden equation

We examine, in more detail, existence results and their accompanying Liouville theorems for the semilinear problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2.77}$$

where, in this section anyway and unless we specify otherwise, we assume U is an open domain, that is, U is an open connected subset of $\mathbb{R}^n$. Specifically, we discuss how the existence results obtained earlier by the calculus of variations are optimal. We will also study how the geometry and topology of the domain influences the existence and non-existence of solutions. For instance, the existence result of Theorem 2.24 is sharp in that the equation admits no classical non-trivial solution in the super-critical case; thus, the only solution is indeed the trivial one. We will later see that this no longer holds if the domain is not star-shaped, e.g., if the domain is an annulus.

Theorem 2.34. *Let $p > (n+2)/(n-2)$ and $U \subset \mathbb{R}^n$ is a bounded open subset with smooth boundary. Further suppose U is a star-shaped domain with respect to the origin. If $u \in C^2(U) \cap C^1(\bar{U})$ is a solution of* (2.77)*, then it must necessarily be the trivial solution $u \equiv 0$.*

For completeness sake, we include the sketch of the proof, which centers on the following Rellich-Pohozaev identity.

Proposition 2.3. *Let $U \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary and star-shaped with respect to the origin. If $u \in C^2(U) \cap C^1(\bar{U})$ is a solution of* (2.77) *with $p > 1$, then*

$$\frac{n-2}{2}\int_U |Du|^2\,dx + \frac{1}{2}\int_{\partial U} |Du|^2(x\cdot\nu)\,dS = \frac{n}{1+p}\int_U |u|^{p+1}\,dx. \tag{2.78}$$

Proof. Multiplying the PDE by $x \cdot Du$ then integrating over U gives us

$$\int_U (x\cdot Du)(-\Delta)u\,dx = \int_U (x\cdot Du)|u|^{p-1}u\,dx.$$

Elementary calculations will show that the left-hand side becomes

$$\int_U (x\cdot Du)(-\Delta)u\,dx = \frac{2-n}{n}\int_U |Du|^2\,dx - \frac{1}{2}\int_{\partial U} |Du|^2(x\cdot\nu)\,dS.$$

Likewise, we calculate that the right-hand term becomes

$$\begin{aligned}\int_U |u|^{p-1}u(x\cdot Du)\,dx &= \frac{1}{p+1}\int_U x\cdot D|u|^{p+1}\,dx \\ &= -\frac{n}{p+1}\int_U |u|^{p+1}\,dx + \frac{1}{p+1}\int_{\partial U}|u|^{p+1}(x\cdot\nu)\,dS \\ &= -\frac{n}{p+1}\int_U |u|^{p+1}\,dx.\end{aligned}$$

The identity follows immediately. □

Proof of Theorem 2.34. Assume otherwise; that is, u is a non-trivial solution of (2.77). If we multiply the PDE by u then integrate over U, we obtain

$$\int_U -u\Delta u\,dx = \int_U |u|^{p+1}\,dx.$$

Then, integration by parts and the zero boundary condition imply that

$$\int_U -u\Delta u\,dx = -\int_{\partial U} u\frac{\partial u}{\partial \nu}\,dS + \int_U |Du|^2\,dx = \int_U |Du|^2\,dx.$$

Hence, we arrive at

$$\int_U |u|^{p+1}\,dx = \int_U |Du|^2\,dx.$$

Inserting this into identity (2.78), we get

$$\Big(\frac{n}{p+1} - \frac{n-2}{2}\Big)\int_U |u|^{p+1}\,dx = \frac{1}{2}\int_{\partial U} |Du|^2(x\cdot\nu)\,dS \geq 0. \tag{2.79}$$

The inequality on the right is due to $x\cdot\nu \geq 0$ on ∂U, since U is star-shaped with respect to the origin. But this implies that $p \leq (n+2)/(n-2)$, which is a contradiction. □

For special domains this non-existence result can be improved to include the borderline critical exponent $p = (n+2)/(n-2)$. For instance, if $U = B_R(0)$ is the ball of radius $R > 0$ centered at the origin, then $x\cdot\nu = R > 0$ on $\partial B_R(0)$. In view of this and Hopf's lemma, if we take $p \geq (n+2)/(n-2)$, then the inequality in (2.79) becomes a strict one. Thus, we can deduce that $p < (n+2)/(n-2)$ and get a contradiction. Hence, combining this with our previous existence result for the subcritical case, we have deduced the following sharp existence result.

Theorem 2.35. *Let $U = B_R(0)$ for any $R > 0$ and $p > 1$. Then equation* (2.46) *admits a classical solution if and only if $p < (n+2)/(n-2)$.*

It is noteworthy to mention that if $U = \mathbb{R}^n$, then the role of the exponent p reverses. Particularly, there holds the following sharp existence result.

Theorem 2.36. *Let $p > 1$ and consider the Lane-Emden equation in the whole space*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \mathbb{R}^n, \\ u > 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{2.80}$$

Then

(a) Equation (2.80) *admits a positive classical solution whenever $p \geq (n+2)/(n-2)$.*

(b) In particular, if $p = (n+2)/(n-2)$, every positive classical solution is radially symmetric and monotone decreasing about some point. Therefore, each positive solution must assume the form

$$u(x) = c_n\Big(\frac{\lambda}{\lambda^2 + |x - x_0|^2}\Big)^{\frac{n-2}{2}}$$

for some constants $c_n, \lambda > 0$ and some point $x_0 \in \mathbb{R}^n$.

(c) Equation (2.80) *has no positive classical solution in the subcritical case, $p < (n+2)/(n-2)$. That is, $u \equiv 0$ is the only non-negative solution of* (2.80)*.*

Proof. In the critical case, the existence of solutions may follow from standard variational methods. In either the super-critical or critical case, the existence of solutions, radially symmetric solutions in particular, follows from a shooting method for ODEs (for a more recent approach combining Brouwer topological fixed point arguments with shooting methods, the reader is referred to [18, 19, 32]). The reason for requiring a shooting method approach is due to the fact that solutions in the super-critical case no longer have finite-energy or belong to a suitable L^p space. Thus, traditional variational methods may no longer apply in this case. Parts (b) and (c) follow from the method of moving planes (see Chapter 5).

□

Besides the critical Sobolev exponent, another critical exponent arises when studying distribution solutions and isolated singularities for the Lane-Emden equation. We define the so-called Serrin exponent p_{se}, where $p_{se} = n/(n-2)$ if $n \geq 3$ and $p_{se} = +\infty$ if $n = 2$. It's interesting to note that the proof of the Liouville theorem for the Lane-Emden equation is substantially simplified in the sub-optimal range, $1 < p < p_{se}$. In fact, the result below holds in the more general class of distributional solutions.

Theorem 2.37. *Let u be a non-negative classical solution of*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \mathbb{R}^n, \\ u > 0 & \text{in } \mathbb{R}^n. \end{cases}$$

Then we necessarily have that $u \equiv 0$.

Proof. Suppose that u is a non-negative entire solution of the Lane-Emden equation. Let $\lambda_1 > 0$ be the first eigenvalue of the Dirichlet Laplacian on the unit ball $B_1(0) \subset \mathbb{R}^n$, and let φ_1 be a corresponding first eigenfunction. We may assume φ_1 is non-negative, $\varphi_1(x) \geq c_0 > 0$ in $B_{1/2}(0)$ and $\varphi_1(x) \leq \varphi_1(0) = \max_{\overline{B}_1(0)} \varphi_1(x) = 1$. If we multiply the equation by $\varphi_R(x) = \varphi_1(R^{-1}x)$, integrate over $B_R(0)$, applying integration by parts and Hölder's inequality leads us to

$$\begin{aligned}\int_{B_R(0)} \varphi_R u^p \, dx &= \int_{B_R(0)} \varphi_R(-\Delta u)\, dx = \int_{\partial B_R(0)} \frac{\partial \varphi_R}{\partial \nu} u \, dS + \int_{B_R(0)} (-\Delta \varphi_R) u \, dx \\ &\leq \frac{\lambda_1}{R^2} \int_{B_R(0)} \varphi_R u \, dx \leq \frac{\lambda_1}{R^2} \Big(\int_{B_R(0)} \varphi_R u^p \, dx \Big)^{1/p} \Big(\int_{B_R(0)} \varphi_R \, dx \Big)^{(p-1)/p} \\ &\leq \frac{\lambda_1}{R^2} |B_R(0)|^{(p-1)/p} \Big(\int_{B_R(0)} \varphi_R \, dx \Big)^{(p-1)/p} \leq (n\omega_n)^{(p-1)/p} \lambda_1 R^{\frac{p-1}{p} - 2},\end{aligned}$$

where we used the fact that $\partial \varphi_R / \partial \nu < 0$ on $\partial B_R(0)$ on the first line, thanks to Hopf's Lemma. This implies that

$$c_0 \int_{B_{R/2}(0)} u^p \, dx \leq \int_{B_R(0)} \varphi_R u^p \, dx \leq n\omega_n \lambda_1^{p/(p-1)} R^{n-2\frac{p}{p-1}}. \tag{2.81}$$

Noting that $p < p_{se}$ implies that $n - 2p/(p-1) < 0$, sending $R \longrightarrow \infty$ in (2.81) shows that $\|u\|_{L^p(\mathbb{R}^n)} = 0$ and thus $u \equiv 0$. □

Now, we examine the properties of solutions of the Lane-Emden equation in the punctured unit ball. We state the results but omit their proofs.

Theorem 2.38. *Let $n \geq 3$ and $1 < p \leq p_S := \frac{n+2}{n-2}$. Assume that u is a positive classical solution of*

$$-\Delta u = u^p \quad \text{in } B_1(0) \backslash \{0\}, \tag{2.82}$$

and that u is unbounded at 0. Then there exist constants $0 < C_1 \leq C_2$ such that

$$C_1 \psi(x) \leq u(x) \leq C_2 \psi(x) \quad \text{for } x \in B_{1/2}(0) \backslash \{0\},$$

where

$$\psi(x) = \begin{cases} |x|^{2-n} & \text{if } 1 < p < p_{se}, \\ |x|^{2-n}(-\log|x|)^{(2-n)/2} & \text{if } p = p_{se}, \\ |x|^{-2/(p-1)} & \text{if } p_{se} < p \leq p_S. \end{cases}$$

The next result is on the removable discontinuity of solutions at the origin.

Theorem 2.39. *Let $p > 1$ and $n \geq 3$. Assume that u is a positive classical solution of (2.82).*

(a) Then $u^p \in L^1_{loc}(B_1(0))$ and there exists $a \geq 0$ such that u is a distribution solution of

$$-\Delta u = u^p + a\delta_0 \quad \text{in } \mathcal{D}'(B_1(0)),$$

where δ_0 denotes the Dirac delta distribution and $\mathcal{D}'(B_1(0))$ is the dual space $\mathcal{D}(B_1(0))$ of test functions on $B_1(0)$. Furthermore, we have $a \leq \bar{a}$ where $\bar{a} = \bar{a}(n,p) > 0$.

(b) If $p < p_{se}$ and $a = 0$, then the singularity is removable, i.e., u is bounded in a neighborhood of the origin.

(c) If $p \geq p_{se}$, then $a = 0$.

Nonlinearly perturbed eigenvalue problems: Calculus of Variations revisted

Consider the more general nonlinear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-1}u & \text{in } U, \\ \quad u = 0 & \text{on } \partial U. \end{cases} \tag{2.83}$$

We have the following non-existence result, which also follows from a Rellich-Pohozaev type identity. We only state the result and omit the proof (but see [27] for the details).

Theorem 2.40. *Let $u \in C^2(U) \cap C^1(\bar{U})$ be a solution of (2.83), $U \subset \mathbb{R}^n$ is a bounded open domain with smooth boundary, and further assume U is a star-shaped domain with respect to the origin.*

(a) If $\lambda < 0$ and $p \geq (n+2)/(n-2)$,

(b) or if $\lambda \leq 0$ and $p > (n+2)/(n-2)$,

then $u \equiv 0$.

To address the question of existence, particularly that of positive solutions, let λ_1 be the first eigenvalue of the Laplace operator $-\Delta$ on $H_0^1(U)$. Recall λ_1 is positive and characterized by the variational formula (see Theorem 2.11)

$$\lambda_1 = \inf_{u \in H_0^1(U),\, u \neq 0} \frac{\displaystyle\int_U |Du|^2\, dx}{\displaystyle\int_U |u|^2\, dx}. \tag{2.84}$$

The next theorem shows that the previous non-existence result is sharp for $\lambda < 0$. In fact, the following existence result remains true for non-negative λ so long as it remains below λ_1.

Theorem 2.41. *Let $1 < p < (n+2)/(n-2)$ and suppose $U \subset \mathbb{R}^n$ is a bounded open domain. Then there exists a positive solution $u \in H_0^1(U)$ to (2.83) provided that $\lambda < \lambda_1$.*

Proof. Consider the functional

$$E(u) = \frac{1}{2}\int_U |Du|^2 - \lambda |u|^2\, dx. \tag{2.85}$$

It suffices to establish the existence of a minimizer for the functional $E(\cdot)$ over the admissible set

$$M = \{u \in H_0^1(U) \mid \|u\|_{L^{p+1}(U)} = 1\}.$$

The proof is the same as that of Theorem 2.24 except that the boundedness from below and the coercivity of the functional need to be verified. Indeed, this is obvious if $\lambda \le 0$. Generally, however, we can easily check that (2.84) implies that

$$E(u) \ge \frac{1}{2}\min\Big\{1, 1-\lambda/\lambda_1\Big\}\|u\|_{H_0^1(U)} \text{ for } u \in H_0^1(U), \text{ whenever } \lambda < \lambda_1.$$

This shows that $E(\cdot)$ is bounded from below and coercive on $H_0^1(U)$. This completes the proof. □

On the other hand, no positive solutions for (2.83) exist in the range $\lambda \ge \lambda_1$.

Theorem 2.42. *Let $U \subset \mathbb{R}^n$ be a bounded open domain, $p > 1$ and suppose $\lambda \ge \lambda_1$. Then problem* (2.83) *does not admit any positive solution in $H_0^1(U)$.*

Proof. We proceed by contradiction. Assume $u \in H_0^1(U)$ is a positive solution of (2.83). Testing the equation in (2.83) by the first eigenfunction $\varphi_1 > 0$ and by integration by parts, we obtain

$$0 = \int_U |u|^{p-1} u\varphi_1\, dx + (\lambda - \lambda_1)\int_U u\varphi\, dx > 0,$$

and we arrive at a contradiction. □

The last existence result for positive solutions can be further refined in the critical case. The following is referred to as the Brezis-Nirenberg theorem, and we state it without proof.

Theorem 2.43 (Brezis-Nirenberg)**.** *Let $p = (n+2)/(n-2)$ and suppose $U \subset \mathbb{R}^n$ is a bounded open domain.*

(a) If $n \ge 4$, there exists a positive solution $u \in H_0^1(U)$ of (2.83) *for any $\lambda \in (0, \lambda_1)$.*

(b) If $n = 3$, there exists $\lambda_* \in [0, \lambda_1)$ such that (2.83) *admits a positive solution $u \in H_0^1(U)$ for each $\lambda \in (\lambda_*, \lambda_1)$.*

(c) If $n = 3$ and $U = B_1(0) \subset \mathbb{R}^3$, then $\lambda_* = \lambda_1/4$ and for $\lambda \le \lambda_*$ there is no positive weak solution to (2.83).

We obtain another existence result for (2.83) with the critical exponent. This next result, albeit a weaker result than the Brezis-Nirenberg Theorem above, illustrates another variational method to obtaining the existence of weak solutions to (2.83) provided the "miminal energy" lies below a threshold determined by the best constant in a Sobolev inequality. The following existence result follows on another variational argument, but prior to stating it, we introduce some notations. We set

$$E_\lambda^U(u) = \frac{\displaystyle\int_U \left(|Du|^2 - \lambda|u|^2\right) dx}{\|u\|^2_{L^{2^*}(U)}} \quad \text{and} \quad S_\lambda(U) = \inf_{u \in H_0^1(U)\setminus\{0\}} E_\lambda^U(u),$$

and the best constant in the Sobolev embedding $H_0^1(U) \hookrightarrow L^{2^*}(U)$ is denoted by S.

Remark 2.22. *The best constant in this Sobolev inequality is discussed in more detail in Chapter 6, particularly Section 6.4. In fact, we will verify that $S = S_\lambda(\mathbb{R}^n) = C_*^{-1}$, where the constant C_* depends only on the dimension n and its explicit form will be calculated in Section 6.4.*

Theorem 2.44. *Let $n \geq 3$, $p = (n+2)/(n-2)$, U is a bounded domain in $\mathbb{R}^n$, and suppose $\lambda > 0$.*

(a) If $S_\lambda(U) < S$, then there exists a function $u \in H_0^1(U)$ such that $u > 0$ in U and $S_\lambda(U) = E_\lambda^U(u)$.

(b) If $0 < \lambda < \lambda_1$ and λ is close to λ_1, then $S_\lambda(U) < S$.

Remark 2.23. *Evidently, $S_\lambda(U) \leq S$ for all $\lambda > 0$ (in fact this holds for all $\lambda \in \mathbb{R}$). Thus, if the minimal energy lies below the sharp Sobolev embedding constant, a minimizer for the constrained variational problem exists. Moreover, part (b) of Theorem* **??** *remains true for all $0 < \lambda < \lambda_1$, but this requires a more delicate analysis using test functions constructed from the one-parameter family of bubble functions for the Lane-Emden equation with critical exponent (these are the essentially unique minimizers for the Sharp Sobolev inequality (see Chapter 6). We will encounter similar problems in Chapter 6 and Section 7.2 in Chapter 7, so we only prove this special case when λ is sufficiently near λ_1.*

Proof of Theorem 2.44. We use the standard argument for constrained minimization problems. **Step 1:** Let $\{u_k\}_{k=1}^\infty$ be a minimizing sequence in $H_0^1(U)$ for S_λ, which we may assume $u_k \geq 0$ and $\|u_k\|_{L^{2^*}(U)} = 1$, for each k. From Hölder's inequality and the boundedness of U, we get $\|u_k\|_{L^2(U)} \leq |U|^{(n+2)/2n}\|u\|_{L^{2^*}(U)} \leq C$ and thus

$$E_\lambda^U(u_k) = \int_U (|Du_k|^2 - \lambda|u_k|^2)\, dx \geq \int_U |Du_k|^2\, dx - \lambda C^2,$$

This shows the minimizing sequence is bounded in $H_0^1(U)$ and by the Rellich-Kondrachov theorem (Theorem A.22), there exists $u \in H_0^1(U)$ such that up to a subsequence, $u_k \rightharpoonup u$ in $H_0^1(U)$, $u_k \longrightarrow u$ strongly in $L^2(U)$, and $u_k \longrightarrow u$ pointwise a.e. in U.

Step 2: We verify u is indeed a minimizer. By Vitali's convergence theorem, Theorem A.29, we have

$$\begin{aligned}\int_U |u_k|^{2^*} - |u_k - u|^{2^*}\,dx &= \int_U \int_0^1 \frac{d}{dt}|u_k + (t-1)u|^{2^*}\,dt\,dx \qquad (2.86)\\ &= 2^* \int_0^1 \int_U (u_k + (t-1)u)|u_k + (t-1)u|^{2^*-2}u\,dx\,dt\\ &\longrightarrow \int_0^1 \int_U tu|tu|^{2^*-2}u\,dx\,dt = \int_U |u|^{2^*}\,dx \text{ as } k \longrightarrow \infty.\end{aligned}$$

We also have

$$\int_U |Du_k|^2\,dx = \int_U |D(u_k - u)|^2\,dx + \int_U |Du|^2\,dx + o(1) \quad \text{as } k \longrightarrow \infty. \qquad (2.87)$$

Hence, applying (2.86) and (2.87) leads us to

$$\begin{aligned}S_\lambda(U) = E^U_\lambda(u_k) + o(1) &= \int_U |D(u_k - u)|^2\,dx + \int_U |Du|^2 - \lambda|u|^2\,dx + o(1)\\ &\geq S\|u_k - u\|^2_{L^{2^*}(U)} + S_\lambda\|u\|^2_{L^{2^*}(U)} + o(1) \geq S\|u_k - u\|^2_{L^{2^*}(U)} + S_\lambda\|u\|^{2^*}_{L^{2^*}(U)} + o(1)\\ &\geq (S - S_\lambda(U))\|u_k - u\|^{2^*}_{L^{2^*}(U)} + S_\lambda(U) + o(1).\end{aligned}$$

Since $S - S_\lambda(U) > 0$, we conclude that $u_k \longrightarrow u$ in $L^{2^*}(U)$. As before, the weakly lower semi-continuity of the norm $\|u\|_{H^1_0(U)} := \|Du\|_{L^2(U)}$ ensures that

$$E^U_\lambda(u) \leq \lim_{k\to\infty} E^U_\lambda(u_k) = S_\lambda(U),$$

and this completes the proof of part (a).

To prove part (b), let $\lambda_1 > 0$ and φ_1 be the first eigenvalue and eigenfunction for the Dirichlet Laplacian in U, and we may assume $\|\varphi_1\|_{L^{2^*}(U)} = 1$. Thus, an integration by parts leads to

$$S_\lambda(U) \leq E^U_\lambda(\varphi_1) = \int_U |D\varphi_1|^2 - \lambda|\varphi_1|^2\,dx = (\lambda_1 - \lambda)\int_U \varphi_1^2\,dx < S,$$

provided λ is sufficiently near λ_1. □

Of course, the minimizers from the previous theorem provides the desired weak solution to problem (2.83).

Corollary 2.1. *Let $n \geq 3$, $p = (n+2)/(n-2)$, U is a bounded domain in $\mathbb{R}^n$, and suppose $0 < \lambda < \lambda_1$. Then* (2.83) *admits a positive weak solution.*

We should point out that the topology of the domain has a direct effect on the solvability of the above elliptic problems. For instance, removing the star-shaped condition on the domain can drastically change the existence of solutions to (2.83). For example, instead let U be the annulus $\{x \in \mathbb{R}^n \,|\, r_1 < |x| < r_2\}$ and consider the Sobolev space of radially symmetric functions

$$H^1_{0,rad}(U) = \{u \in H^1_0(U) \,|\, u(x) = u(|x|)\}.$$

Since U is an annulus, the key point here is that the embedding $H^1_{0,rad}(U) \hookrightarrow L^{p+1}(U)$ remains compact for all $p > 1$! So we may apply a variational method with constraint or use a mountain pass approach on $E(\cdot)$ within this class of radial functions. Thus, we can show the existence of infinitely-many radially symmetric positive solutions to (2.83) for any $1 < p < \infty$ and $\lambda \in \mathbb{R}$.

Remark 2.24. *In each of the existence results in this section, the assumption that solutions belong to $C^2(U) \cap C^1(\bar{U})$ can be replaced with the weaker assumption that solutions belong to $H^1_0(U)$. This is due to the regularity theory for weak solutions, which we cover in the next chapter.*

2.12.2 A Doubling Lemma and A Priori Estimates

The previous Liouville theorems are key to obtaining a priori estimates for closely related Dirichlet problems, which are themselves important ingredients in obtaining existence and regularity results. To get such a priori bounds, one way is to assume such bounds do not hold. Then a blow-up or rescaling argument can be used to eventually reach a contradiction with a Liouville theorem. We prove the following basic result to illustrate how to carry out this idea.

Theorem 2.45. *Let $n \geq 3$ and $1 < p < (n+2)/(n-2)$, and suppose $U \subset \mathbb{R}^n$ is a proper domain of $\mathbb{R}^n$. Then there exists a universal positive constant $C = C(n,p)$, independent of U and u, such that any non-negative classical solution u of $\Delta u + u^p = 0$ in U satisfies*

$$u(x) + |Du(x)|^{2/(p+1)} \leq C dist(x, \partial U)^{-2/(p-1)} \text{ for } x \in U. \tag{2.88}$$

In particular, if U is an exterior domain, i.e., it contains the set $\{x \in \mathbb{R}^n \,|\, |x| > R\}$ for some $R > 0$, then

$$u(x) + |Du(x)|^{2/(p+1)} \leq C|x|^{-2/(p-1)}, \; |x| \geq 2R.$$

The proof of Theorem 2.45 will require the following doubling lemma.

Lemma 2.9 (Doubling)**.** *Let (X, d) be a complete metric space, and let $D \subset \Sigma \subset X$ with D non-empty and Σ closed. Set $\Gamma = \Sigma \backslash D$, fix a real number k, and assume $M : D \mapsto (0, \infty)$ is bounded on compact subsets of D. Then, if $y \in D$ such that*

$$M(y) dist(y, \Gamma) > 2k, \tag{2.89}$$

then there exists $x \in D$ such that

$$M(x) dist(x, \Gamma) > 2k, \; M(x) \geq M(y), \tag{2.90}$$

and

$$M(z) \le 2M(x) \text{ for all } z \in D \cap \bar{B}_X(x, k/M(x)). \tag{2.91}$$

Remark 2.25. *The version of the doubling lemma we presented above is far more abstract than what we actually need. In particular, we will take $X = \mathbb{R}^n$, U an open subset of $\mathbb{R}^n$ and we set $D = U$ and $\Sigma = \bar{D}$. Then $\Gamma = \partial U$ and we have $\bar{B}_X(x, k/M(x)) \subset D$. Moreover, since D is open,* (2.90) *implies that*

$$dist(x, D^c) = dist(x, \Gamma) > 2k/M(x).$$

Proof of the Doubling Lemma. We proceed by contradiction. Assuming the lemma is not valid, we claim there exists a sequence $\{x_j\}$ in D such that

$$M(x_j)dist(x_j, \Gamma) > 2k, \tag{2.92}$$

$$M(x_{j+1}) > 2M(x_j) \tag{2.93}$$

and

$$d(x_j, x_{j+1}) \le k/M(x_j) \tag{2.94}$$

for $j = 0, 1, 2, \ldots$. Choose $x_0 = y$. By our contradiction argument assumption, there exists $x_1 \in D$ such that

$$M(x_1) \ge 2M(x_0)$$

and

$$d(x_0, x_1) \le k/M(x_0).$$

Fix some integer $i \ge 1$ and assume that we have already constructed $x_0, \ldots, x_i$ so that (2.92)–(2.94) hold for $j = 0, 1, 2, \ldots, i-1$. Hence,

$$dist(x_i, \Gamma) \ge dist(x_{i-1}, \Gamma) - d(x_{i-1}, x_i) > (2k - k)/M(x_{i-1}) > 2k/M(x_i),$$

and so

$$M(x_i)dist(x_i, \Gamma) > 2k.$$

Since we also have $M(x_i) \ge M(y)$, our contradiction assumption implies there exists $x_{i+1} \in D$ such that

$$M(x_{i+1}) > 2M(x_i)$$

and

$$d(x_i, x_{i+1}) \le k/M(x_i).$$

This proves the claim by induction. Therefore, we have that

$$M(x_i) \ge 2^i M(x_0) \text{ and } d(x_i, x_{i+1}) \le k2^{-i}M(x_0)^{-1} \text{ for } i = 0, 1, 2, \ldots. \tag{2.95}$$

Namely, if $\{x_i\}$ is a Cauchy sequence, then it converges to some point $a \in \bar{D} \subset \Sigma$. Moreover,

$$d(x_0, x_i) \leq \sum_{j=0}^{i-1} d(x_j, x_{j+1}) \leq kM(x_0)^{-1} \sum_{j=0}^{i-1} 2^{-j} \leq 2kM(x_0)^{-1},$$

and thus

$$dist(x_i, \Gamma) \geq dist(x_0, \Gamma) - 2kM(x_0)^{-1} > 0.$$

Therefore,

$$K = \{x_0, x_1, x_2, \ldots, \} \cup \{a\}$$

is a compact subset of $D = \Sigma \backslash \Gamma$. Now, since (2.95) implies that $M(x_i) \longrightarrow \infty$ as $i \longrightarrow \infty$, we see that M is unbounded on the compact subset $K \subset D$. Hence, we deduce a contradiction with the boundedness of M on compact subsets of D, and this completes the proof.

□

Proof of Theorem 2.9. Assume estimate (2.88) fails. Then there exist sequences

$$U_k, u_k, \text{ and } y_k \in U_k, \quad \text{for } k = 0, 1, 2, \ldots$$

such that u_k solves $\Delta u_k + u_k^p = 0$ in U_k and the functions

$$M_k(x) = u_k(x)^{(p-1)/2} + |Du_k(x)|^{(p-1)/(p+1)} \tag{2.96}$$

satisfy

$$M_k(y_k) > 2k/dist(y_k, \partial U_k). \tag{2.97}$$

By the Doubling Lemma and our previous remark, there exists $x_k \in U_k$ such that

$$M_k(x_k) \geq M_k(y_k), \quad M_k(x_k) > 2k/dist(x_k, \partial U_k),$$

and

$$M_k(z) \leq 2M_k(x_k), \; |z - x_k| \leq k/M_k(x_k).$$

Now we rescale u_k by setting

$$v_k(y) = \lambda_k^{2/(p-1)} u_k(x_k + \lambda_k y), \; |y| \leq k$$

with

$$\lambda_k = M_k(x_k)^{-1}$$

so that v_k satisfies the elliptic equation

$$\Delta_y v_k(y) + v_k(y)^p = 0, \; |y| \leq k.$$

Moreover,

$$v_k(0)^{(p-1)/2} + |Dv_k(0)|^{(p-1)/(p+1)} = \lambda_k M_k(x_k) = 1, \tag{2.98}$$

and

$$v_k(y)^{(p-1)/2} + |Dv_k(y)|^{(p-1)/(p+1)} \leq 2, \ \ |y| \leq k.$$

By standard elliptic L^p estimates and Sobolev embeddings, we can show there exists a subsequence of $\{v_k\}$ that converges to some v in $C^1_{loc}(\mathbb{R}^n)$, which is a non-negative classical solution of $\Delta v + v^p = 0$ in $\mathbb{R}^n$.

Furthermore, (2.98) ensures that

$$v(0)^{(p-1)/2} + |Dv(0)|^{(p-1)/(p+1)} = 1,$$

which implies v is non-trivial. This contradicts part (c) of Theorem 2.36, i.e., it contradicts the fact that $v \equiv 0$ is the only such non-negative entire solution for the subcritical Lane-Emden equation. This completes the proof.

□

CHAPTER 3

Regularity Theory for Second-order Elliptic Equations

This chapter compiles the basic regularity theory for second-order elliptic equations in divergence form, i.e., elliptic equations of the type

$$Lu = f \quad \text{in } U,$$

where U is a bounded open subset of $\mathbb{R}^n$ and L admits the form in (1.2).

Basically, we may classify the study of regularity properties of solutions into three main types:

(A) Schauder's approach or the regularity theory for classical solutions

(B) Calderón-Zygmund or L^p theory

(C) Hölder regularity of weak solutions (using both perturbation and iteration approaches)

Our goal is to cover elementary regularity results along with their proofs for each type, but we must prepare some background material beforehand.

3.1 Preliminaries

In this section, we provide a concise treatment of the tools we require in establishing various regularity results for elliptic equations. Namely, we study the weak L^p, BMO and Morrey–Campanato spaces, the Calderón–Zygmund Decomposition, and the Marcinkiewicz interpolation inequalities.

3.1.1 Flattening out the Boundary

We often assume that the boundary of our domain U is smooth in some sense in order to establish regularity estimates at the boundary. Roughly speaking, such assumptions allows us to flatten the boundary locally and treat it much like what we would do in establishing interior regularity estimates. In particular, let U be an open and bounded domain in $\mathbb{R}^n$ and $k \in \{1, 2, 3, \ldots\}$.

Definition 3.1. *We say the boundary ∂U is C^k if for each point $x^0 \in \partial U$ there exists $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ such that, upon relabeling and reorienting the coordinate axes if necessary, we have*

$$U \cap B_r(x^0) = \{x \in B_r(x^0) \,|\, x_n > \gamma(x_1, \ldots, x_{n-1})\}.$$

Likewise, we say ∂U is C^∞ if ∂U is C^k for each $k = 1, 2, 3, \ldots,$ and we say ∂U is analytic if the mapping γ is analytic.

We often need to change the coordinates near a boundary point of ∂U as to flatten out the boundary. More precisely, fix $x^0 \in \partial U$ and choose γ and r as in the previous definition. Define $y_i = x_i =: \Phi^i(x)$ if $i = 1, 2, \ldots, n-1$ and $y_n = x_n - \gamma(x_1, \ldots, x_{n-1}) =: \Phi^n(x)$, and write

$$y = \Phi(x).$$

Similarly, we set $x_i = y_i =: \Psi^i(y)$ for $i = 1, 2, \ldots, n-1$ and $x_n = y_n + \gamma(y_1, \ldots, y_{n-1}) =: \Psi^n(y)$, and write

$$x = \Psi(y).$$

Then

$$\Phi = \Psi^{-1}$$

and the mapping $x \mapsto \Phi(x) = y$ "straightens out" the boundary ∂U near x^0. Observe additionally that these maps are volume preserving, i.e.,

$$\det D\Phi = \det D\Psi = 1.$$

3.1.2 Weak Lebesgue Spaces and Lorentz Spaces

Let X, or more precisely $(X, \mathcal{A}, \mu)$, be a measure space where μ is a positive, not necessarily finite, measure on X. In most cases, we take $X = \mathbb{R}^n$ with the usual n-dimensional Lebesgue measure. For a measurable function f on X, the **distribution function** of f is the function d_f defined on $[0, \infty)$ as follows:

$$d_f(t) = \mu(\{x \in X : |f(x)| > t\}).$$

Some basic properties of distribution functions are given by the following proposition.

Proposition 3.1. *Let f and g be measurable functions on X. Then for all $s, t > 0$ we have*

(a) $|g| \leq |f|$ *μ–a.e. implies that* $d_g \leq d_f$,

(b) $d_{cf}(t) = d_f(t/|c|)$ *for all* $c \in \mathbb{C}\backslash\{0\}$,

(c) $d_{f+g}(s+t) \leq d_f(s) + d_g(t)$,

(d) $d_{fg}(st) \leq d_f(s) + d_g(t)$.

Now we describe L^p norm in terms of the distribution function and define the weak L^p space.

Proposition 3.2. *For $f \in L^p(X)$, $0 < p < \infty$, we have*

$$\|f\|_{L^p}^p = p \int_0^\infty t^{p-1} d_f(t)\, dt.$$

Proof.

$$\begin{aligned} p \int_0^\infty t^{p-1} d_f(t)\, dt &= p \int_0^\infty t^{p-1} \int_X \chi_{\{x \in X : |f(x)| > t\}}\, d\mu(x)\, dt \\ &= \int_X \int_0^{|f(x)|} p t^{p-1}\, dt\, d\mu(x) \\ &= \int_X |f(x)|^p\, d\mu(x) \\ &= \|f\|_{L^p}^p, \end{aligned}$$

where we used Fubini's Theorem in the second equality. □

Definition 3.2. *For $0 < p < \infty$, the space weak $L^p(X)$, also denoted by $L^p_w(X)$ or $L^{p,\infty}(X)$, is defined as the set of all μ-measurable functions f such that*

$$\begin{aligned} \|f\|_{L^{p,\infty}} &= \inf \left\{ C > 0 : d_f(t) \leq \left(\frac{C}{t} \right)^p \quad \text{for all} \quad t > 0 \right\} \\ &= \sup \left\{ t d_f(t)^{1/p} : t > 0 \right\} \end{aligned}$$

is finite. The space weak $L^\infty(X)$ is by definition $L^\infty(X)$.

Remark 3.1. *The weak $L^p(X)$ space is commonly denoted by $L^p_w(X)$ or by its equivalent Lorentz space characterization $L^{p,\infty}(X)$. Moreover, we can show that*

(a) $\|f\|_{L^{p,\infty}} = 0 \Longrightarrow f = 0$ *μ a.e.*,

(b) $\|kf\|_{L^{p,\infty}} = |k| \|f\|_{L^{p,\infty}}$,

(c) $\|f+g\|_{L^{p,\infty}} \le \max\{2, 2^{1/p}\}(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}})$.

Hence, the triangle inequality does not hold so that $L^{p,\infty}(X)$ is a quasi-normed linear space for $0 < p < \infty$. In fact, these spaces are complete.

Obviously, the weak L^p spaces are larger than L^p spaces.

Proposition 3.3. *For any $0 < p < \infty$ and any $f \in L^p(X)$, we have*

$$\|f\|_{L^{p,\infty}} \le \|f\|_{L^p}.$$

Hence, $L^p(X) \hookrightarrow L^{p,\infty}(X)$.

Proof. This is a trivial consequence of Chebyshev's inequality:

$$t^p d_f(t) \le \int_{\{x\in X\,:\,|f(x)|>t\}} |f(x)|^p \, d\mu(x).$$

□

Definition 3.3. *An operator $T : L^p(X) \longrightarrow L^q(X)$ is of* **strong type** (p,q) *if*

$$\|Tf\|_{L^q} \le C\|f\|_{L^p} \quad \text{for all} \quad f \in L^p(X).$$

Similarly, T is of **weak type** (p,q) *if*

$$\|Tf\|_{L^{q,\infty}} \le C\|f\|_{L^p} \quad \text{for all} \quad f \in L^p(X).$$

For completeness, we introduce the Lorentz spaces in which the Lebesgue and weak Lebesgue spaces are special cases. First, if f is a real (or complex) valued function defined on X, then the **decreasing rearrangement** of f is the function f^* defined on $[0,\infty)$ by

$$f^*(t) = \inf\{s > 0 \,|\, d_f(s) \le t\}.$$

We adopt the convention that $\inf \emptyset = \infty$, thus $f^*(t) = \infty$ whenever $d_f(s) > t$ for all $s \ge 0$. Now, given a measurable function f on X and $0 < p, q \le \infty$, define

$$\|f\|_{L^{p,q}(X)} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \, \frac{dt}{t} \right)^{1/q}$$

whenever $q < \infty$, and if $q = \infty$ we take

$$\|f\|_{L^{p,\infty}(X)} = \sup_{t>0} t^{1/p} f^*(t).$$

Then the set of all f with $\|f\|_{L^{p,q}(X)} < \infty$ is denoted by $L^{p,q}(X)$ and is called the **Lorentz space** with indices p and q. It is interesting to note several properties of the decreasing rearrangement of f. Namely, we have that

(a) $d_f = d_{f^*}$,

(b) $(|f|^p)^* = (f^*)^p$ whenever $0 < p < \infty$,

(c) $\int_X |f|^p \, d\mu = \int_0^\infty f^*(t)^p \, dt$ whenever $0 < p < \infty$,

(d) $\sup_{t>0} t^q f^*(t) = \sup_{\alpha>0} \alpha (d_f(\alpha))^q$ for $0 < q < \infty$.

In view of these properties, it is simple to verify that $L^{p,p}(X) = L^p(X)$, $L^{\infty,\infty}(X) = L^\infty(X)$, and weak $L^p(X) = L^{p,\infty}(X)$.

3.1.3 The Marcinkiewicz Interpolation Inequalities

The following is known as the Marcinkiewicz interpolation theorem. A more general "non-diagonal" version involving the Lorentz spaces holds, but we shall not make use of it in these notes and thus omit it.

Theorem 3.1 (Marcinkiewicz interpolation)**.** *Let T be a linear operator from $L^p(X) \cap L^q(X)$ into itself with $1 \le p < q \le \infty$. If T is of weak type (p,p) and weak type (q,q), then for any $p < r < q$, T is of strong type (r,r). More precisely, if there exist constants B_p and B_q such that*

$$d_{Tf}(t) \le \left(\frac{B_p \|f\|_{L^p}}{t} \right)^p$$

and

$$d_{Tf}(t) \le \left(\frac{B_q \|f\|_{L^q}}{t} \right)^q$$

for all $f \in L^p(X) \cap L^q(X)$, then

$$\|Tf\|_{L^r} \le C B_p^\theta B_q^{1-\theta} \|f\|_r \quad \text{for all} \quad f \in L^p(X) \cap L^q(X),$$

where

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

and $C = C(p,q,r)$ is a positive constant. In fact,

$$C(p,q,r) = 2\left(\frac{r}{r-p} + \frac{r}{q-r} \right)^{1/r}.$$

Note that if $q = \infty$, then the $L^q(X)$ and $L^{q,\infty}(X)$ spaces and their norms above are replaced with the space $L^\infty(X) = L^{\infty,\infty}(X)$ and its norm.

3.1.4 Calderón–Zygmund and the John-Nirenberg Lemmas

Lemma 3.1 (Calderón–Zygmund Decomposition)**.** *For $f \in L^1(\mathbb{R}^n)$, a fixed $\alpha > 0$, there exists E and G such that*

(a) $\mathbb{R}^n = E \cup G$, $E \cap G = \emptyset$,

(b) $|f(x)| \leq \alpha$ *a.e.* $x \in E$,

(c) $G = \cup_{k=1}^{\infty} Q_k$, $\{Q_k\}$ *are disjoint cubes for which*

$$\alpha < \frac{1}{|Q_k|} \int_{Q_k} |f(x)|\, dx \leq 2^n \alpha.$$

Lemma 3.2 (John-Nirenberg)**.** *Suppose $u \in L^1(U)$ satisfies*

$$\int_{B_r(x)} |u - (u)_{x,r}|\, dy \leq M r^n \quad \textit{for any} \quad B_r(x) \subset U.$$

Then there holds for any $B_r(x) \subset U$

$$\int_{B_r(x)} e^{\frac{p_0}{M}|u-(u)_{x,r}|}\, dy \leq C r^n$$

for some positive p_0 and C depending only on n.

3.1.5 L^p Boundedness of Integral Operators

We briefly introduce some basic results on integral operators of convolution type but our goal is to ultimately prove the Hardy-Littlewood-Sobolev (HLS) inequality. However, we will need some basic properties of the Hardy-Littlewood maximal function in order to prove the HLS inequality. The weak Lebesgue spaces, the Calderón–Zygmund decomposition and the Marcinkiewicz interpolation inequalities will play very important roles here.

Specifically, the operators we consider are examples of singular integral operators whose kernels do not belong to a proper L^p space but rather to a weak L^p space, e.g., the Riesz type kernel $|x|^{-(n-\alpha)}$ belongs to $L^{p,\infty}(\mathbb{R}^n)$ but not to $L^p(\mathbb{R}^n)$ when $p = n/(n-\alpha)$. This type of issue is relevant in the L^p regularity theory for elliptic partial differential equations studied later in this chapter. Particularly, we shall see in Section 3.2 that deriving $W^{2,p}$ a priori estimates on weak solutions requires showing certain differential operators involving the Newtonian potentials are weak and strong type operators. A similar dichotomy appears for the maximal function operators.

The function

$$\mathcal{M}(f)(x) = \sup_{\delta>0} Avg_{B_\delta(x)}|f| = \sup_{\delta>0} \frac{n}{\omega_n \delta^n} \int_{B_\delta(0)} |f(x-y)|\, dy$$

is called the **centered Hardy-Littlewood maximal function** of f. Likewise, the function

$$M(f)(x) = \sup_{\delta>0,\,|z-x|<\delta} Avg_{B_\delta(z)}|f|$$

is called the **uncentered Hardy-Littlewood maximal function** of f.

Clearly, $\mathcal{M}(f) \leq M(f)$. Also, $\mathcal{M}(f) = \mathcal{M}(|f|) \geq 0$, i.e., the maximal function is a positive operator, and obviously $\mathcal{M}$ maps $L^\infty(\mathbb{R}^n)$ to itself, i.e.,

$$\|\mathcal{M}(f)\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^\infty(\mathbb{R}^n)}.$$

We show that the maximal function as an integral operator is of weak type $(1,1)$ and thus is of strong type (p,p) for any $1 < p < \infty$ by interpolation. The proof of this requires the following basic result which is sometimes referred to as the Vitali covering lemma.

Lemma 3.3 (Vitali Covering)**.** *Let $\{B_1, B_2, \ldots, B_k\}$ be a finite collection of open balls in $\mathbb{R}^n$. Then there exists a finite subcollection $\{B_{j_1}, B_{j_2}, \ldots, B_{j_\ell}\}$ of pairwise disjoint balls such that*

$$\sum_{r=1}^{\ell} |B_{j_r}| \geq 3^{-n}\Big|\bigcup_{i=1}^{k} B_i\Big|. \tag{3.1}$$

Proof. Without loss of generality, we can assume that the collection of balls satisfies

$$|B_1| \geq |B_2| \geq \ldots \geq |B_k|.$$

Let $j_1 = 1$. Having chosen $j_1, j_2, \ldots, j_i$, let j_{i+1} be the least index $s > j_i$ such that $\cup_{m=1}^{i} B_{j_m}$ is disjoint from B_s. Since we have a finite collection of balls, this process must stop after some ℓ finite number of steps. Indeed, this yields a finite subcollection of pairwise disjoint balls $B_{j_1}, B_{j_2}, \ldots, B_{j_\ell}$. If some B_m was not selected, i.e., $m \notin \{j_1, j_2, \ldots, j_\ell\}$, then B_m must intersect a selected ball B_{j_r} for some $j_r < m$. Then B_m has smaller size than B_{j_r} and we must have $B_m \subseteq 3B_{j_r}$. This shows that the union of the unselected balls is contained in the union of triples of the selected balls. Thus, the union of all balls is contained in the union of the triples of the selected balls and so

$$\Big|\bigcup_{i=1}^{k} B_i\Big| \leq |\bigcup_{r=1}^{\ell} 3B_{j_r}| \leq \sum_{r=1}^{\ell} |3B_{j_r}| = 3^n \sum_{r=1}^{\ell} |B_{j_r}|.$$

This completes the proof. □

Theorem 3.2. *The uncentered Hardy-Littlewood maximal function maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ with constant at most 3^n and also $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$ with constant at most $3^{n/p}p(p-1)^{-1}$. The same is true for the centered maximal operator $\mathcal{M}$.*

Proof. Since $M(f) \geq \mathcal{M}(f)$, we have

$$\{x \in \mathbb{R}^n \,|\, |\mathcal{M}(f)(x)| > t\} \subseteq \{x \in \mathbb{R}^n \,|\, |M(f)(x)| > t\},$$

and therefore it suffices to show that

$$d_{M(f)}(t) := |\{x \in \mathbb{R}^n \,|\, |M(f)(x)| > t\}| \leq 3^n \frac{\|f\|_{L^1(\mathbb{R}^n)}}{t}. \tag{3.2}$$

Step 1: We claim that the set

$$E_t = \left\{x \in \mathbb{R}^n \,\middle|\, |M(f)(x)| > t\right\}$$

is an open subset of $\mathbb{R}^n$. Indeed, for $x \in E_t$ there is an open ball B_x containing x such that the average of $|f|$ over B_x is strictly bigger than t. Then the uncentered maximal function of any other point in B_x is also bigger than t, and thus B_x is contained in E_t. This proves that E_t is open.

Step 2: Estimate (3.2) holds.

Let K be any compact subset of E_t. For each $x \in K$ there exists an open ball B_x containing the point x such that

$$\int_{B_x} |f(y)|\, dy > t|B_x|. \tag{3.3}$$

Observe that $B_x \subset E_t$ for all x, and by compactness there exists a finite subcover

$$\{B_{x_1}, B_{x_2}, \ldots, B_{x_k}\} \text{ of the subset } K.$$

In view of Lemma 3.3, we find a subcollection of pairwise disjoint balls $B_{x_{j_1}}, \ldots, B_{x_{j_\ell}}$ such that (3.1) holds and combining this with (3.3) yields

$$|K| \leq \left|\bigcup_{i=1}^{k} B_{x_i}\right| \leq 3^n \sum_{i=1}^{\ell} |B_{x_{j_i}}| \leq \frac{3^n}{t} \sum_{i=1}^{\ell} \int_{B_{x_{j_i}}} |f(y)|\, dy \leq \frac{3^n}{t} \int_{E_t} |f(y)|\, dy$$

since all the balls $B_{x_{j_i}}$ are disjoint and contained in E_t. From this we deduce (3.2) after taking the supremum over all compact subsets of $K \subseteq E_t$ and using the inner regularity of the Lebesgue measure. This verifies $M = M(f)$ (as well as $\mathcal{M} = \mathcal{M}(f)$) is of weak type $(1, 1)$. Recall that M is of strong type (p, p) with $p = \infty$. Thus, the Marcinkiewicz interpolation theorem (see Theorem 3.1) implies the operator M is of strong type (p, p) for all $1 < p < \infty$ and that

$$\|M(f)\|_{L^p} \leq C\|f\|_{L^p},$$

where $C = 2(\frac{p}{p-1})^{1/p} 3^{n/p}$. As indicated in the theorem, we may improve this bound with the slightly sharper constant $C = 3^{n/p} \frac{p}{p-1}$ but we leave this to the reader to verify (see Exercise 1.3.3 in [14]). Likewise, the same argument applies to the centered Hardy-Littlewood maximal function, $\mathcal{M}(f)$. This completes the proof of the theorem. $\square$

The following result states that the maximal operator controls the averages of a function with respect to any radially decreasing integrable function. We omit the proof but refer to Theorem 2.1.10 in [14].

Theorem 3.3. *Let $k \geq 0$ be a function on $[0, \infty)$ that is continuous except at a finite number of points. Suppose that $K(x) = k(|x|)$ is an integrable function on $\mathbb{R}^n$ and satisfies*

$$K(x) \geq K(y) \ \text{ whenever } \ |x| \leq |y|,$$

i.e., k is decreasing. Then

$$\sup_{\epsilon>0} |f| * K_\epsilon(x) \leq \|K\|_{L^1(\mathbb{R}^n)} \mathcal{M}(f)(x)$$

for all locally integrable functions f on $\mathbb{R}^n$. Here $K_\epsilon(x) = \epsilon^{-n} K(x/\epsilon)$. An important case is when $K(x) = |x|^{\alpha-n} \chi_{|x|<R}(x)$ for any fixed $R \in (0, \infty)$ and $\alpha \in (0, n)$.

With the results presented above, we are now ready to offer some important applications of the Hardy-Littlewood maximal functions.

The Lebesgue Differentiation Theorem

Theorem 3.4 (Lebesgue Differentiation Theorem)**.** *For any $f \in L^1_{loc}(\mathbb{R}^n)$, there holds*

$$\lim_{r \longrightarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)\, dy = f(x) \ \text{ for a.e. } x \in \mathbb{R}^n. \tag{3.4}$$

Consequently, $|f| \leq \mathcal{M}(f)$ almost everywhere.

Before we prove this, we need some preliminary tools. First, let (X, μ) and (Y, ν) be two measure spaces, $p \in (0, \infty]$ and $q \in (0, \infty)$. Suppose that D is a dense subspace of $L^p(X, \mu)$ and for every $\epsilon > 0$, T_ϵ is a linear operator on $L^p(X, \mu)$ with values in the set of measurable functions on Y. Define the sublinear operator

$$T_*(f)(x) = \sup_{\epsilon>0} |T_\epsilon(f)(x)|.$$

Theorem 3.5. *Let $p, q \in (0, \infty)$. Suppose that for some constant $C > 0$ and all $f \in L^p(X, \mu)$ we have*

$$\|T_*(f)\|_{L^{q,\infty}} \leq C\|f\|_{L^p},$$

and for all $f \in D$,

$$\lim_{\epsilon \longrightarrow 0} T_\epsilon(f) = T(f) \tag{3.5}$$

exists and is finite for ν-a.e. and defines a linear operator on D. Then, for all $f \in L^p(X, \mu)$, the limit (3.5) exists and finite ν-a.e. and uniquely defines an operator T on $L^p(X, \mu)$, by the continuous extension of T on the dense subspace D, such that

$$\|T(f)\|_{L^{q,\infty}} \leq C\|f\|_{L^p}. \tag{3.6}$$

Proof. Given $f \in L^p(X, \mu)$, we define the oscillation of f by

$$O_f(y) = \limsup_{\epsilon \to 0} \limsup_{\theta \to 0} |T_\epsilon(f)(y) - T_\theta(f)(y)|.$$

We claim that for all $f \in L^p(X, \mu)$ and $\delta > 0$,

$$\nu(\{y \in Y \mid O_f(y) > \delta\}) = 0. \tag{3.7}$$

Once, we prove this claim, then $O_f(y) = 0$ for ν-a.e. y, which further implies that $T_\epsilon(f)(y)$ is Cauchy for ν-a.e. y. This implies that $T_\epsilon(f)(y)$ converges ν-a.e. to some $T(f)(y)$ as $\epsilon \longrightarrow 0$. The operator T defined this way on $L^p(X, \mu)$ is linear and extends T defined on D.

We now prove the claim. Choose $\eta > 0$ and by density, we may choose $g \in D$ such that $\|f - g\|_{L^p} < \eta$. Since $T_\epsilon(g) \longrightarrow T(g)$ ν-a.e., it follows that $O_g = 0$ ν-a.e. From this and the linearity of T_ϵ, we conclude that

$$O_f(y) \leq O_g(y) + O_{f-g}(y) = O_{f-g}(y) \text{ for } \nu\text{-a.e. } y.$$

Now for any $\delta > 0$, we have

$$\begin{aligned}\nu(\{y \in Y \mid O_f(y) > \delta\}) &\leq \nu(\{y \in Y \mid O_{f-g}(y) > \delta\}) \\ &\leq \nu(\{y \in Y \mid 2T_*(f-g)(y) > \delta\}) \\ &\leq ((2C/\delta)\|f-g\|_{L^p})^q \leq (2C\eta/\delta)^q.\end{aligned}$$

Then sending $\eta \longrightarrow 0$, we deduce (3.7). We thus conclude that $T_\epsilon(f)$ is a Cauchy sequence and hence converges ν-a.e. to some $T(f)$. Since $|T(f)| \leq |T_*(f)|$, the estimate (3.6) follows immediately. □

Proof of Theorem 3.4. Since $\mathbb{R}^n$ is locally compact and is the union of the open balls $B_N(0)$, $N = 1, 2, 3, \ldots$, it suffices to prove the theorem for almost every x inside the ball $B_N(0)$. Then we may take f supported in a larger ball, thus working with f integrable over the whole space $\mathbb{R}^n$.

Let $T_\epsilon(f) = K_\epsilon * f$, where $K_\epsilon(x) = \epsilon^{-n}k(x/\epsilon)$ with $k = |B_1(0)|^{-1}\chi_{B_1(0)}$. We know that the corresponding operator T_* is controlled by the centered Hardy-Littlewood maximal function $\mathcal{M}$ (see Theorem 3.3), which maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, i.e., $\mathcal{M}$ is an operator of weak type $(1, 1)$. Hence, T_* must also be of weak type $(1, 1)$.

It is easy to show that (3.4) holds in the space of continuous functions f with compact support, which is dense in $L^1(\mathbb{R}^n)$. From this and the fact that T_* maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, Theorem 3.5 implies that (3.4) holds for all $f \in L^1(\mathbb{R}^n)$. □

The Hardy-Littlewood-Sobolev inequality

Consider the integral operator

$$I_\alpha(f)(x) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

first introduced in Definition 1.3. We stated–without proof–the boundedness of the operator I_α in Lebesgue spaces in Theorem 1.25. We are now prepared to prove this so-called Hardy-Littlewood-Sobolev (HLS) inequality. The proof that we present here centers on the strong boundedness of the Hardy-Littlewood maximal function and Theorem 3.3.

Theorem 3.6 (HLS inequality)**.** *Let $\alpha \in (0, n)$ and $1 < p < q < \infty$ satisfy*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}.$$

Then there exists a finite positive constant $C = C(n, \alpha, p)$ such that for all $f \in L^p(\mathbb{R}^n)$ there holds

$$\|I_\alpha(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.8}$$

Proof. The main idea is to estimate the operator I_α in terms of the Hardy-Littlewood maximal function. Specifically, our estimates below will involve the uncentered maximal operator $M(f)$. First, observe that $I_\alpha(f)$ is well-defined in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ which is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. So it suffices to assume that $f \in \mathcal{S}(\mathbb{R}^n)$. We may also assume that $f \geq 0$ since $I_\alpha(|f|) \geq |I_\alpha(f)|$. Now consider the splitting,

$$\int_{\mathbb{R}^n} f(x-y)|y|^{\alpha-n}\,dy = J_1(f)(x) + J_2(f)(x),$$

where

$$\begin{aligned} J_1(f)(x) &= \int_{|y|<R} f(x-y)|y|^{\alpha-n}\,dy, \\ J_2(f)(x) &= \int_{|y|\geq R} f(x-y)|y|^{\alpha-n}\,dy, \end{aligned}$$

and $R > 0$ is some constant to be specified below.

Estimating J_1: Particularly, J_1 is given by convolution with the function $|y|^{\alpha-n}\chi_{|y|<R}$. So by applying Theorem 3.3, we have that

$$J_1(f)(x) \leq M(f)(x)\int_{|y|<R}|y|^{\alpha-n}\,dy = \frac{\omega_n}{\alpha}R^\alpha M(f)(x).$$

Estimating J_2: Hölder's inequality yields

$$\begin{aligned} |J_2(f)(x)| &\leq \Big(\int_{|y|\geq R}|y|^{p(\alpha-n)/(p-1)}\,dy\Big)^{(p-1)/p}\|f\|_{L^p(\mathbb{R}^n)} \\ &= \Big(\frac{(p-1)q\omega_n}{pn}\Big)^{(p-1)/p}R^{-n/q}\|f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Combining the above estimates for J_1 and J_2 yields for any $R > 0$

$$I_\alpha(f)(x) \le C(n,\alpha,p)(R^\alpha M(f)(x) + R^{-n/q}\|f\|_{L^p(\mathbb{R}^n)}).$$

Hence, by choosing a constant multiple of the quantity

$$R = \|f\|_{L^p(\mathbb{R}^n)}^{p/n}(M(f)(x))^{-p/n},$$

we reduce the previous estimate to

$$I_\alpha(f)(x) \le C(n,\alpha,p)M(f)(x)^{p/q}\|f\|_{L^p(\mathbb{R}^n)}^{1-p/q}. \tag{3.9}$$

We deduce the desired result by raising estimate (3.9) to the power q, integrating over $\mathbb{R}^n$ then using the fact that $M(f)$ is of strong type (p,p) for any $1 < p < \infty$ (see Theorem 3.2). This completes the proof. □

Remark 3.2. *Interestingly enough, a weaker version of the HLS inequality holds in the endpoint case $p = 1$ but with the original estimate* (3.8) *being replaced with the estimate*

$$\|I_\alpha(f)\|_{L^{q,\infty}(\mathbb{R}^n)} \le C(n,\alpha)\|f\|_{L^1(\mathbb{R}^n)}$$

where $q = n/(n-\alpha)$. The proof of this is just as before since the weaker inequality will follow from the estimate (3.9) *and the fact that $M(f)$ is of weak type* $(1,1)$.

Fractional Sobolev inequalities from the HLS inequality

We shall derive fractional Sobolev inequalities in $\mathbb{R}^n$ as a result of the HLS inequality. We refer the readers to the appendix A and the indicated references for a brief review of Sobolev spaces and their embedding properties. We shall provide a more in-depth study of a sharp Sobolev inequality in Chapter 6.

Theorem 3.7 ($W^{s,p}$ Sobolev Embedding)**.** *Let $s \in (0,n)$ and suppose $sp < n$. We define the Sobolev space $W^{s,p}(\mathbb{R}^n)$ as the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm*

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |(-\Delta)^{s/2}f|^p\,dx\Big)^{1/p} \simeq \Big(\int_{\mathbb{R}^n} (|\xi|^s|\widehat{f}(\xi)|)^p\,d\xi\Big)^{1/p}.$$

Then $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, where $q = np/(n-sp)$, and there exists a positive constant C, depending only on n, p, and s such that

$$\|f\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{W^{s,p}(\mathbb{R}^n)} \quad \text{for all } f \in W^{s,p}(\mathbb{R}^n).$$

Proof. Choose any $f \in W^{s,p}(\mathbb{R}^n)$. Then $g = (-\Delta)^{s/2}f$ belongs to $L^p(\mathbb{R}^n)$ and using the properties of the Riesz potentials, we may write $f = I_s(g)$. Since

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{p} - \frac{n-sp}{np} = \frac{sp}{np} = \frac{s}{n},$$

the HLS inequality implies

$$\|f\|_{L^q(\mathbb{R}^n)} = \|I_s g\|_{L^q(\mathbb{R}^n)} \le C\|g\|_{L^p(\mathbb{R}^n)} = C\|(-\Delta)^{s/2}f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{W^{s,p}(\mathbb{R}^n)}.$$

for some positive constant $C = C(n,p,s)$. This completes the proof. □

The Hilbert and Riesz Transforms

For completeness, we look at another prototypical example of a singular integral operator of convolution type called the Hilbert transform. There are several ways to define the Hilbert transform. First, we give its definition as a convolution operator with a certain principle value distribution. We begin by defining the distribution $W_0 \in \mathcal{S}'(\mathbb{R})$ as

$$\langle W_0, \varphi \rangle = \pi^{-1} \lim_{\epsilon \to 0} \int_{\epsilon \leq |x| \leq 1} \frac{\varphi(x)}{x} \, dx + \pi^{-1} \int_{|x| \geq 1} \frac{\varphi(x)}{x} \, dx \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}).$$

Then the Hilbert transform of $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$H(f)(x) = (W_0 * f)(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(x-y)}{y} \, dy = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} \, dy, \tag{3.10}$$

where

$$P.V. \int_{-\infty}^{\infty} F(x,y) \, dy = \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} F(x,y) \, dy$$

is the usual principle value integral.

Remark 3.3. *Note that*

$$\int_{-\infty}^{\infty} \frac{f(x-y)}{y} \, dy$$

does not converge absolutely, and it is important to notice that the function $1/y$ integrated over $[-1, -\epsilon] \cap [\epsilon, 1]$ has mean value 0. Therefore, this is precisely why we must treat the above improper integral in the principal value sense. Also, for each $x \in \mathbb{R}$, $H(f)(x)$ is defined for all integrable functions f on $\mathbb{R}$ that satisfy a Hölder condition near the point x.

Alternatively, we can define the Hilbert transform using the Fourier transform. Namely, there holds

$$\widehat{W_0}(\xi) = -isgn(\xi),$$

and so

$$H(f)(x) = \mathcal{F}^{-1}(\widehat{f}(\xi)[-isgn(\xi)])(x). \tag{3.11}$$

An immediate consequence of (3.11) is that H is an isometry on $L^2(\mathbb{R})$, i.e.,

$$\|H(f)\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

Moreover, it follows that the adjoint of H is $H^* = -H$. Now, as with the Hardy-Littlewood maximal operator, the Hilbert transform is of strong type (p,p) for all $1 < p < \infty$. We sketch the proof of this. First, we can show the estimate

$$|\{x \in \mathbb{R} \, | \, |H(\chi_E)(x)| > t\}| \leq \frac{2}{\pi} \frac{|E|}{t}, \quad t > 0,$$

holds for all subsets E of the real line of finite measure. This inequality and a basic result (see Theorem 1.4.19 in [14]) ensure H is bounded on $L^p(\mathbb{R})$ for $1 < p < 2$. By duality, $H^* = -H$ is bounded on $L^p(\mathbb{R})$ for $2 < p < \infty$. Thus, H is also bounded on $L^p(\mathbb{R})$ for $2 < p < \infty$. Finally, H is an isometry on $L^2(\mathbb{R})$. This completes the proof.

The Riesz tranforms are the n-dimensional analogue of the Hilbert transform. To introduce such transforms, we introduce the tempered distributions W_j on $\mathbb{R}^n$, for $1 \leq j \leq n$ as follows. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, let

$$\langle W_j, \varphi \rangle = \frac{\Gamma(\frac{n+2}{2})}{\pi^{\frac{n+1}{2}}} P.V. \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} \varphi(y)\, dy.$$

Then the **jth Riesz transform** of f, denoted by $R_j(f)$, is given by convolution with W_j, i.e.,

$$R_j(f)(x) = (f * W_j)(x) = \frac{\Gamma(\frac{n+2}{2})}{\pi^{\frac{n+1}{2}}} P.V. \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Alternatively, the jth Riesz transform can be defined via the Fourier transform, i.e.,

$$R_j(f)(x) = \mathcal{F}^{-1}(-\frac{i\xi_j}{|\xi|}\widehat{f}(\xi))(x) \text{ for all } f \in \mathcal{S}(\mathbb{R}^n).$$

Interestingly enough, the Riesz transforms satisfy

$$-Identity = \sum_{j=1}^{n} R_j^2.$$

Likewise, the jth Riesz transforms R_j are bounded operators on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Application of Riesz tranforms to the Poisson equation

Another interesting application of Riesz tranforms is to Poisson's equation. Namely, suppose that f belongs to $\mathcal{S}(\mathbb{R}^n)$ and u is a tempered distribution that solves the elliptic equation

$$-\Delta u = f.$$

Indeed, there holds from the Fourier transform that

$$(4\pi^2|\xi|^2)\widehat{u}(\xi) = \widehat{f}(\xi).$$

Notice that for all $1 \leq j, k \leq n$ we have

$$\partial_j\partial_k u = \mathcal{F}^{-1}((2\pi i\xi_j)(2\pi i\xi_k)\widehat{u}(\xi)) = \mathcal{F}^{-1}\Big((2\pi i\xi_j)(2\pi i\xi_k)\frac{\widehat{f}(\xi)}{4\pi^2|\xi|^2}\Big) = R_jR_k(f) = R_jR_k(-\Delta u).$$

That is, we conclude that $\partial_j\partial_k u$ are functions. Thus, Riesz transforms provide an explicit way to recover second-order derivatives in terms of the Laplacian.

Remark 3.4. *If $f = 0$, then we reduce the problem to the Laplace equation, $\Delta u = 0$, and a solution $u \in \mathcal{S}'(\mathbb{R}^n)$ is usually called a* ***harmonic distribution****. As above, applying the Fourier transform yields $\widehat{\Delta u} = 0$ and so*

$$-4\pi^2|\xi|^2\widehat{u} = 0 \ \ in \ \ \mathcal{S}'(\mathbb{R}^n).$$

This implies that $\widehat{u}$ is supported at the origin, so applying the inverse Fourier transform implies the Liovuille theorem that u is a polynomial.

3.2 $W^{2,p}$ Regularity for Weak Solutions

This section covers the L^p or so-called Calderón-Zygmund regularity theory for second-order elliptic equations.

3.2.1 $W^{2,p}$ A Priori Estimates

Initially, we will establish the $W^{2,p}$ a priori estimates for the Newtonian potentials, then extend the result to general elliptic equations.

Theorem 3.8 ($W^{2,p}$ a priori Estimate for the Newtonian Potential)**.** *Let $f \in L^p(U)$ for $1 < p < \infty$, and let $w = \Gamma * f$ be the Newtonian potential of f. Then $w \in W^{2,p}(U)$ and*

$$-\Delta w = f(x) \quad a.e.\ x \in U \quad and \quad \|D^2 w\|_{L^p} \leq C\|f\|_{L^p}.$$

Proof. We provide a sketch of the proof in four key steps. We define the linear operator T by

$$Tf = D_{ij}\Gamma * f.$$

Observe that it suffices to show that T is a bounded linear operator on $L^p(U)$.
Step 1: $T : L^2(U) \longrightarrow L^2(U)$ is a bounded linear operator, i.e., T is of strong type $(2,2)$.

Let $f \in C_0^\infty(U) \subset C_0^\infty(\mathbb{R}^n)$. Recall that $w \in C^\infty(\mathbb{R}^n)$ and satisfies Poisson's equation

$$-\Delta w = f(x) \ \text{ in } \ \mathbb{R}^n.$$

With the help of the Fourier transform and Plancherel's identity,

$$\begin{aligned}
\int_U |f(x)|^2\,dx &= \int_{\mathbb{R}^n} |f(x)|^2\,dx = \int_{\mathbb{R}^n} |\Delta w|^2\,dx = \int_{\mathbb{R}^n} |\widehat{\Delta w(\xi)}|^2\,d\xi \\
&= \int_{\mathbb{R}^n} |\xi|^4|\widehat{w(\xi)}|^2\,dx = \sum_{k,j=1}^n \int_{\mathbb{R}^n} \xi_k^2\xi_j^2|\widehat{w(\xi)}|^2\,d\xi \\
&= \sum_{k,j=1}^n \int_{\mathbb{R}^n} |\widehat{D_{kj}w(\xi)}|^2\,d\xi = \sum_{k,j=1}^n \int_{\mathbb{R}^n} |D_{kj}w(x)|^2\,dx \\
&= \int_{\mathbb{R}^n} |D^2 w|^2\,dx.
\end{aligned}$$

Hence, $\|Tf\|_{L^2} \leq \|f\|_{L^2}$ for all $f \in C_0^\infty(U)$ and so $T : L^2(U) \longrightarrow L^2(U)$ is a bounded linear operator simply by the density of $C_0^\infty(U)$ in $L^2(U)$.

Step 2: T is of weak type $(1,1)$.

This result follows from the Calderón–Zygmund decomposition and we skip its proof for the sake of brevity, but the reader is referred to [6][page 82] for the proof.

Step 3: T is of strong type (p,p) for any $1 < p < \infty$.

Since T is of weak type $(1,1)$ and is of strong type $(2,2)$—therefore is of weak type $(2,2)$—the Marcinkiewicz interpolation theorem implies that T is of strong type (r,r) for $1 < r \leq 2$. Given any $2 < q < \infty$, let $r = \frac{q}{q-1} \in (1,2]$. By duality and the fact that T is of strong type (r,r), we see that

$$\begin{aligned}
\|Tf\|_{L^q} &= \sup_{\|g\|_{L^r}=1} \langle g, Tf\rangle := \sup_{\|g\|_{L^r}=1} \int_U g(x)Tf(x)\,dx \\
&= \sup_{\|g\|_{L^r}=1} \langle Tg, f\rangle \leq \sup_{\|g\|_{L^r}=1} \|Tg\|_{L^r}\|f\|_{L^q} \\
&\leq \sup_{\|g\|_{L^r}=1} C_r\|g\|_{L^r}\|f\|_{L^q} \\
&\leq C_r\|f\|_{L^q}.
\end{aligned}$$

Thus, T is of strong type (q,q) for $q \in (2,\infty)$. Hence, T is of strong type (p,p) for any $1 < p < \infty$. □

Now we present the $W^{2,p}$ a priori estimates on strong solutions for the uniformly elliptic equation with bounded coefficients:

$$Lu = f(x) \quad \text{in} \quad U. \tag{3.12}$$

Definition 3.4. *We say that u is a* **strong solution** *of* (3.12) *if u is twice weakly differentiable in U and satisfies the equation almost everywhere in U.*

Throughout this section, we assume $U \subset \mathbb{R}^n$ is bounded and open with $C^{2,\alpha}$ boundary, $a^{ij} \in C(\bar{U})$, $b^i \in L^q(U)$ and $c \in L^q(U)$ for some $q \in (n,\infty]$. In the details below, we will assume $q = \infty$ for simplicity.

Theorem 3.9 ($W^{2,p}$ Estimates for Uniformly Elliptic Equations)**.** *Let $1 < p < \infty$, $f \in L^p(U)$, and let $u \in W^{2,p}(U) \cap H_0^1(U)$ be a strong solution of* (3.12)*. Then*

$$\|u\|_{W^{2,p}} \leq C\left(\|u\|_{L^p} + \|f\|_{L^p}\right)$$

where $C = C(\lambda, \Lambda, n, p, U, \|b_i\|_{L^\infty}, \|c\|_{L^\infty})$ is a positive constant.

Proof. The proof can be separated into two major estimates—the interior estimate and the boundary estimate.

Part I: Interior Estimate

$$\|D^2u\|_{L^p(K)} \leq C\left(\|Du\|_{L^p(U)} + \|u\|_{L^p(U)} + \|f\|_{L^p(U)}\right) \tag{3.13}$$

where K is any compact subset of U.

Part II: Boundary Estimate

$$\|D^2u\|_{L^p(U\setminus U_\delta)} \leq C\left(\|Du\|_{L^p(U)} + \|u\|_{L^p(U)} + \|f\|_{L^p(U)}\right) \tag{3.14}$$

where $U_\delta = \{x \in U \,|\, dist(x, \partial U) > \delta\}$.

Part III: The interior and boundary estimates imply

$$\|u\|_{W^{2,p}(U)} \leq C\left(\|u\|_{L^p(U)} + \|f\|_{L^p(U)}\right). \tag{3.15}$$

To see this, it is obvious that both estimates yield

$$\begin{aligned}\|u\|_{W^{2,p}(U)} &\leq \|u\|_{W^{2,p}(U\setminus U_{2\delta})} + \|u\|_{W^{2,p}(U_\delta)}\\ &\leq C\left(\|Du\|_{L^p(U)} + \|u\|_{L^p(U)} + \|f\|_{L^p(U)}\right).\end{aligned} \tag{3.16}$$

We have the following estimate

$$\begin{aligned}\|Du\|_{L^p(U)} &\leq C\|u\|^{1/2}_{L^p(U)}\|D^2u\|^{1/2}_{L^p(U)}\\ &\leq \epsilon\|D^2u\|_{L^p(U)} + \frac{C}{4\epsilon}\|u\|_{L^p(U)}\end{aligned}$$

where the first inequality is the well-known Gagliardo–John–Nirenberg interpolation inequality and the second inequality is the basic Cauchy inequality with ϵ. Substituting this into (3.16) yields

$$\|u\|_{W^{2,p}(U)} \leq C\epsilon\|D^2u\|_{L^p(U)} + C\left(\frac{C}{4\epsilon}\|u\|_{L^p(U)} + \|u\|_{L^p(U)} + \|f\|_{L^p(U)}\right).$$

If we choose $\epsilon < \frac{1}{2C}$, we can absorb the $C\epsilon\|D^2u\|_{L^p(U)}$ term on the right-hand side by the left-hand side and arrive at the desired estimate. □

Let us give provide the details in obtaining interior and boundary estimates.

Part I: Interior Estimates We proceed using the well-known method of frozen coefficients. Define the cut-off function $\varphi \in C_c^\infty(\mathbb{R})$ to be the function

$$\varphi(s) := \begin{cases} 1 & \text{if } s \leq 1,\\ 0 & \text{if } s \geq 2.\end{cases}$$

Then we measure the *module* continuity of the coefficients a^{ij} with

$$\epsilon(\delta) = \sup_{|x-y|\le\delta, x,y\in U, 1\le i,j\le n} |a^{ij}(x) - a^{ij}(y)|.$$

Note that the function $\epsilon(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$. Then for any $x_0 \in U_{2\delta}$, let

$$\eta(x) = \varphi\left(\frac{|x-x_0|}{\delta}\right) \quad \text{and} \quad w(x) = \eta(x)u(x).$$

We compute

$$\begin{aligned}
a^{ij}(x_0)\frac{\partial^2 w}{\partial x_i \partial x_j} &= (a^{ij}(x_0) - a^{ij}(x))\frac{\partial^2 w}{\partial x_i \partial x_j} + a^{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} \\
&= (a^{ij}(x_0) - a^{ij}(x))\frac{\partial^2 w}{\partial x_i \partial x_j} + \eta(x)a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} \\
&\quad + a^{ij}(x)u(x)\frac{\partial^2 \eta}{\partial x_i \partial x_j} + 2a^{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_j} \\
&= (a^{ij}(x_0) - a^{ij}(x))\frac{\partial^2 w}{\partial x_i \partial x_j} + \eta(x)\left(b^i(x)\frac{\partial u}{\partial x_i} + c(x)u - f(x)\right) \\
&\quad + a^{ij}(x)u(x)\frac{\partial^2 \eta}{\partial x_i \partial x_j} + 2a^{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_j} \\
&:= -F(x) \quad \text{for} \quad x \in \mathbb{R}^n.
\end{aligned}$$

Notice that all terms in F are supported in $B_{2\delta}(x_0) \subset U$. By the uniformly elliptic condition, we can assume $a^{ij}(x_0) = \delta_{ij}$ by a simple linear transformation. Thus, w and $\Gamma * F$ both satisfy the problem of $-\Delta u = F$ in $\mathbb{R}^n$, where u and F are both compactly supported in $\mathbb{R}^n$. Thus, the uniqueness property for this problem implies $w \equiv \Gamma * F$. Then, by our earlier estimates on the Newtonian potential, we obtain

$$\|D^2 w\|_{L^p(B_{2\delta}(x_0))} = \|D^2 w\|_{L^p(\mathbb{R}^n)} \le C\|F\|_{L^p(\mathbb{R}^n)} = C\|F\|_{L^p(B_{2\delta}(x_0))}. \tag{3.17}$$

Estimating each term in F yields

$$\|F\|_{L^p(B_{2\delta}(x_0))} \le \epsilon(2\delta)\|D^2 w\|_{L^p(B_{2\delta}(x_0))} + \|f\|_{L^p(B_{2\delta}(x_0))} + C\left(\|Du\|_{L^p(B_{2\delta}(x_0))} + \|u\|_{L^p(B_{2\delta}(x_0))}\right).$$

Combining this estimate with the estimate (3.17) and choosing δ sufficiently small so that $C\epsilon(2\delta) < 1/2$, we have

$$\|D^2 w\|_{L^p(B_{2\delta}(x_0))} \le \frac{1}{2}\|D^2 w\|_{L^p(B_{2\delta}(x_0))} + C\left(\|f\|_{L^p(B_{2\delta}(x_0))} + \|Du\|_{L^p(B_{2\delta}(x_0))} + \|u\|_{L^p(B_{2\delta}(x_0))}\right),$$

which is equivalent to

$$\|D^2 w\|_{L^p(B_{2\delta}(x_0))} \le C\left(\|f\|_{L^p(B_{2\delta}(x_0))} + \|Du\|_{L^p(B_{2\delta}(x_0))} + \|u\|_{L^p(B_{2\delta}(x_0))}\right).$$

Hence,

$$\|D^2u\|_{L^p(B_\delta(x_0))} \le C\left(\|f\|_{L^p(B_{2\delta}(x_0))} + \|Du\|_{L^p(B_{2\delta}(x_0))} + \|u\|_{L^p(B_{2\delta}(x_0))}\right),$$

where we used the fact that $\|D^2u\|_{L^p(B_\delta(x_0))} = \|D^2w\|_{L^p(B_\delta(x_0))}$ since $u \equiv w$ on $B_\delta(x_0)$.

We can easily extend this estimate from a δ-ball to any compact subset K of U via a standard covering argument. Namely, for any compact subset $K \subset U$, let $\delta < \frac{1}{2}dist(K, \partial U)$, then $K \subset U_{2\delta}$ and we can derive the desired interior estimate:

$$\|D^2u\|_{L^p(K)} \le C\left(\|f\|_{L^p(U)} + \|Du\|_{L^p(U)} + \|u\|_{L^p(U)}\right).$$

Part II: Boundary Estimates.

The main ideas used in establishing the boundary estimate are relatively similar to the proof of the interior estimate. Roughly speaking, we may flatten out the boundary and treat the regularity problem as one on an upper half-space. We refer the reader to [6, 7, 13] for more details and we only sketch the main steps here. More precisely, for any point $x_0 \in \partial U$, the intersection $B_\delta(x_0) \cap \partial U$ is a $C^{2,\alpha}$ graph for $\delta > 0$ small enough. Therefore, after flattening out the boundary, we may assume that this graph is given by

$$x_n = h(x_1, x_2, \ldots, x_{n-1}) = h(x'),$$

and U lies on top of this graph locally. Now let $y = \psi(x) = (x' - x_0', x_n - h(x'))$ so that ψ is a diffeomorphism mapping a neighborhood of x_0 onto the upper ball $B_r^+(0) = \{y \in B_r(0) \,|\, y_n > 0\}$. Under this map, the elliptic equation becomes

$$\begin{cases} -\bar{a}^{ij}(y)D_{ij}u(y) + \bar{b}_i(y)D_iu(y) + \bar{c}(y)u(y) = \bar{f}(y) & \text{in } B_r^+(0), \\ u(y) = 0 & \text{on } \partial B_r^+(0). \end{cases} \tag{3.18}$$

Here the coefficients come from the original coefficients under the diffeomorphism ψ. For example, using the chain rule,

$$\bar{a}^{ij}(y) = \frac{\partial \psi^i}{\partial x^\ell}(\psi^{-1}(y))a^{\ell k}(\psi^{-1}(y))\frac{\partial \psi^j}{\partial x^k}(\psi^{-1}(y)).$$

We can assume $\bar{a}^{ij}(0) = \delta_{ij}$ otherwise we can apply a linear transformation to ensure this property holds. Moreover, since planes are mapped to planes under this diffeomorphism, we can assume problem (3.18) is valid even for smaller r. Applying the method of frozen coefficients with $w(y) = \varphi(2|y|/r)u(y)$ yields

$$-\Delta w(y) = F(y) \text{ in } B_r^+(0).$$

Now let $\bar{w}(y)$ and $\bar{F}(y)$, respectively, be the odd extension of $w(y)$ and $F(y)$ from $B_r^+(0)$ to $B_r(0)$. More precisely,

$$\bar{w}(y) := \begin{cases} w(y_1, y_2, \ldots, y_{n-1}, y_n) & \text{if } y_n \geq 0, \\ -w(y_1, y_2, \ldots, y_{n-1}, -y_n) & \text{if } y_n < 0. \end{cases}$$

and

$$\bar{F}(y) := \begin{cases} F(y_1, y_2, \ldots, y_{n-1}, y_n) & \text{if } y_n \geq 0, \\ -F(y_1, y_2, \ldots, y_{n-1}, -y_n) & \text{if } y_n < 0. \end{cases}$$

We can show that

$$-\Delta \bar{w}(y) = \bar{F}(y) \text{ in } B_r(0).$$

Thus, we can apply the same arguments as before to get the basic interior estimate for this problem, i.e.,

$$\begin{aligned} \|D^2 u\|_{L^p(B_r(x_0))} &\leq C(\|f\|_{L^p(B_{2r}(x_0))} + \|Du\|_{L^p(B_{2r}(x_0))} + \|u\|_{L^p(B_{2r}(x_0))}) \\ &\leq C(\|f\|_{L^p(B_{2r}(x_0)\cap U)} + \|Du\|_{L^p(B_{2r}(x_0)\cap U)} + \|u\|_{L^p(B_{2r}(x_0)\cap U)}), \end{aligned}$$

and this holds for any x_0 on ∂U and for some small radius $r > 0$. Note that the last line of the previous estimate follows from the symmetric extension of w to $\bar{w}$ from the half ball to the whole ball.

Furthermore, these balls form a covering of the boundary ∂U. By compactness of this boundary, there is a finite cover $B_{r_i}(x_i)$, $i = 1, 2, \ldots k$. These balls also cover a neighborhood of ∂U including $U \backslash U_\delta$ for some suitably small $\delta > 0$. Summing the estimates over each ball in the finite cover will imply the desired boundary estimate

$$\|u\|_{W^{2,p}(U\backslash U_\delta)} \leq C(\|Du\|_{L^p(U)} + \|u\|_{L^p(U)} + \|f\|_{L^p(U)}).$$

This completes the proof of the $W^{2,p}$ a priori estimates.

3.2.2 Regularity of Solutions and A Priori Estimates

Let $1 < p < \infty$. So far, we have established a priori estimates to solutions in the $W^{2,p}(U)$ norm by assuming weak solutions were already strong solutions belonging to $H_0^1(U) \cap W^{2,p}(U)$. Here we shall only assume u is a weak solution in $W_0^{1,p}(U)$. Then we actually show that u necessarily belongs to $W^{2,p}(U)$ with the help of the a priori estimates. The procedure for doing so has many points in common with our earlier derivations of the $W^{2,p}$ a priori estimates but with some subtle differences.

We say $u \in W_0^{1,p}(U)$ is a weak solution of

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{3.19}$$

if for any $v \in W_0^{1,q}(U)$ with $1/p + 1/q = 1$,

$$\int_U \left[a^{ij}(x) D_i u D_j v + b^i(x)(D_i u)v + c(x)uv\right] dx = \int_U f(x)v\,dx.$$

Although this notion of weak solution relies on duality to define the equation in the distribution sense, the density of $C_0^\infty(U)$ in $W_0^{1,q}(U)$ ensures it is enough for the identity to hold for all test functions $v \in C_0^\infty(U)$. Our main result is the following.

Theorem 3.10. *Let $n \geq 2$ and $1 < p < \infty$ and let $U \subset \mathbb{R}^n$ be a bounded and open subset. Suppose that L is a uniformly elliptic operator whose leading coefficient $a^{ij}(x)$ is Lipschitz continuous in U, and the lower-order terms $b^i(x)$ and $c(x)$ are bounded functions in U. If $u \in W_0^{1,p}(U)$ is a weak solution of the boundary value problem* (3.19) *where $f \in L^p(U)$, then $u \in W^{2,p}(U)$.*

We shall see that the uniqueness of weak solutions of (3.19) is an important ingredient in establishing our regularity result. We only consider the case $p \geq 2$ since the uniqueness of solutions is simpler in this situation. The reason is that the uniqueness of weak solutions will allow us to improve the a priori estimates.

Lemma 3.4. *Assume that if $u \in W^{1,p}(U)$ is a weak solution of*

$$Lu = f(x) \text{ in } U, \tag{3.20}$$

then the a priori estimate

$$\|u\|_{W^{2,p}(U)} \leq C(\|u\|_{L^p(U)} + \|f\|_{L^p(U)})$$

holds. In addition, assume uniqueness holds in the sense that if $Lu = 0$, then $u \equiv 0$ in U. Then, for the unique solution u of (3.20)*, we obtain the refined a priori estimate*

$$\|u\|_{W^{2,p}(U)} \leq C\|f\|_{L^p(U)}. \tag{3.21}$$

Proof. Assume the inequality (3.21) is false. That is, there exists a sequence of functions (f_k) with $\|f_k\|_{L^p(U)} = 1$ and the sequence of corresponding solutions (u_k) satisfying

$$Lu_k = f_k(x) \text{ in } U,$$

such that

$$\|u_k\|_{W^{2,p}(U)} \longrightarrow \infty \text{ as } k \longrightarrow \infty.$$

We consider the normalized functions

$$v_k := u_k/\|u_k\|_{L^p(U)} \text{ and } g_k := f_k/\|u_k\|_{L^p(U)}.$$

Thus,

$$\|v_k\|_{L^p(U)} = 1 \text{ and } \|g_k\|_{L^p(U)} \longrightarrow 0 \text{ as } k \longrightarrow \infty, \tag{3.22}$$

and

$$Lv_k = g_k(x) \text{ in } U. \tag{3.23}$$

Of course, we have the a priori estimate

$$\|v_k\|_{W^{2,p}(U)} \le C(\|v_k\|_{L^p(U)} + \|g_k\|_{L^p(U)}).$$

Combining this with (3.22) shows (v_k) is bounded in $W^{2,p}(U)$ and so the Banach-Alaoglu theorem implies there exists a subsequence, which we still label as (v_k), that converges weakly to some $v \in W^{2,p}(U)$. On the other hand, the compact Sobolev embedding implies that the same subsequence converges strongly to $v \in L^p(U)$, and hence $\|v\|_{L^p(U)} = 1$. Sending $k \longrightarrow \infty$ in (3.23) shows

$$Lv = 0 \text{ in } U.$$

By the uniqueness assumption, $v \equiv 0$, but this contradicts with $\|v\|_{L^p(U)} = 1$. This completes the proof. □

Proposition 3.4. *Let $p \ge 1$ and assume $f \in L^p(B_1(0))$. Then the Dirichlet problem*

$$\begin{cases} -\Delta u = f & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0), \end{cases} \tag{3.24}$$

has a unique solution $u \in W^{2,p}(B_1(0))$ satisfying

$$\|u\|_{W^{2,p}(B_1(0))} \le C\|f\|_{L^p(B_1(0))}. \tag{3.25}$$

Proof. Uniqueness follows by testing the equation against u, integrating over $B_1(0)$ then integrating by parts to get

$$\int_{B_1(0)} |Du|^2 \, dx = 0.$$

Thus, $Du \equiv 0$ and so u is constant in $B_1(0)$. The boundary condition further implies that $u \equiv 0$.

Since f is continuous, the existence of solutions follows from the integral representation,

$$u(x) = \int_{B_1(0)} G(x,y) f(y) \, dy, \; x \in B_1(0),$$

where $G(x,y)$ is the Green's function for the region $B_1(0)$. More precisely,

$$G(x,y) = \Gamma(y-x) - \phi^x(y)$$

where $\Gamma(x)$ is the fundamental solution of Laplace's equation and $\phi^x(y)$, when $n \ge 3$, is the corrector function (c.f., (1.60))

$$\phi^x(y) = \frac{1}{(n-2)\omega_n}(|x||x/|x|^2 - y|)^{2-n}.$$

It remains to show the $W^{2,p}$ estimate for this integral representation of the solution. Of course, we have already established the estimate for the first part

$$\int_{B_1(0)} \Gamma(x-y) f(y)\, dy$$

since this is just the Newtonian potential of $f(x)$, but we are missing the estimate for the part involving the corrector function. Instead, we proceed with an approximation argument. For $\delta > 0$ suitably small, consider the ball $B_{1-\delta}(0)$ and set

$$u_\delta(x) = \int_{B_1(0)} G(x,y) f_\delta(y)\, dy,$$

where

$$f_\delta(x) := \begin{cases} f(x) & \text{if } x \in B_{1-\delta}(0), \\ 0 & \text{elsewhere.} \end{cases}$$

From our earlier result on Newtonian potentials, there holds that $D^2 u_\delta$ belongs to $L^p(B_1(0))$. Thus, by Poincaré's inequality, u_δ belongs to $L^p(B_1(0))$ and hence, to $W^{2,p}(B_1(0))$ as well. From Lemma 3.4, we have the improved a priori estimate

$$\|u_\delta\|_{W^{2,p}(B_1(0))} \le C \|f_\delta\|_{L^p(B_1(0))}.$$

We may choose a sequence $\{\delta_i\} \longrightarrow 0^+$ so that the corresponding solutions $\{u_{\delta_i}\}$ is a Cauchy sequence in $W^{2,p}(B_1(0))$. This follows since

$$\|u_{\delta_i} - u_{\delta_j}\|_{W^{2,p}(B_1(0))} \le C \|f_{\delta_i} - f_{\delta_j}\|_{L^p(B_1(0))} \longrightarrow 0$$

as $i, j \longrightarrow \infty$. Then let u_0 be the limit point of this Cauchy sequence in $W^{2,p}(B_1(0))$. Then $u_0 \in W^{2,p}(B_1(0))$ is a solution of (3.24) and the improved a priori estimate (3.25) holds. This completes the proof.

□

Proof of Theorem 3.10. In view of our comments above, assume that $p \ge 2$. Consider the usual smooth cut-off function

$$\varphi(s) := \begin{cases} 1 & \text{if } s \le 1, \\ 0 & \text{if } s \ge 2. \end{cases}$$

Let $u \in W_0^{1,p}(U)$ be a weak solution of (3.19). For any x_0 in $U_{2\delta} := \{x \in U \,|\, dist(x, \partial U) \ge 2\delta\}$, let

$$\eta(x) = \varphi\Big(\frac{|x-x_0|}{\delta}\Big) \text{ and } w(x) = \eta(x) u(x).$$

Thus, w is supported in $B_{2\delta}(x_0)$. By our definition of a weak solution in $W_0^{1,p}(U)$, it is easily verified that for any $v \in C_0^\infty(B_{2\delta}(x_0))$,

$$\int_{B_{2\delta}(x_0)} a^{ij}(x_0) D_i w D_j v\, dx = \int_{B_{2\delta}(x_0)} [a^{ij}(x_0) - a^{ij}(x)] D_i w D_j v + F(x) v\, dx,$$

where

$$F(x) = f(x) - D_j(a^{ij}(x)(D_i\eta)u) - b^i(x)D_iu - c(x)u.$$

Namely, w is a weak solution of

$$\begin{cases} -a^{ij}(x_0)D_{ij}w = -D_j([a^{ij}(x_0) - a^{ij}(x)]D_iw) + F(x) & \text{in } B_{2\delta}(x_0), \\ w = 0 & \text{on } \partial B_{2\delta}(x_0). \end{cases} \tag{3.26}$$

As before, we may assume $a^{ij}(x_0) = \delta_{ij}$ and we may rewrite (3.26) as

$$\begin{aligned} -\Delta w = & - D_j([a^{ij}(x_0) - a^{ij}(x)]D_iw) + F(x) \\ = & - [a^{ij}(x_0) - a^{ij}(x)]D_{ij}w + \tilde{F}(x) \text{ in } B_{2\delta}(x_0), \end{aligned} \tag{3.27}$$

where

$$\tilde{F}(x) = D_j[a^{ij}(x)]D_iw + F(x).$$

For any $v \in W^{2,p}(B_{2\delta}(x_0))$, clearly

$$[a^{ij}(x_0) - a^{ij}(x)]D_{ij}v \in L^p(B_{2\delta}(x_0)).$$

In addition, it is easy to verify that $\tilde{F}$ belongs to $L^p(B_{2\delta}(x_0))$. In view of Proposition 3.4, the Laplacian Δ is an invertible linear operator, and so we may consider the equation

$$v = Kv + (-\Delta)^{-1}\tilde{F} \quad \text{in } W^{2,p}, \tag{3.28}$$

where

$$Kv(x) := \Delta^{-1}([a^{ij}(x_0) - a^{ij}(x)]D_{ij}v).$$

From the Lipschitz continuity of $a^{ij}(x)$, K is a contraction mapping from $W^{2,p}(B_{2\delta}(x_0))$ to itself provided that $\delta > 0$ is sufficiently small. Thus, there exists a unique solution $v \in W^{2,p}(B_{2\delta}(x_0))$ to equation (3.28). By the uniqueness of solutions of (3.27), which follows from arguments similar to those in the proof of Proposition 3.4, we have that $w \equiv v$ in $W^{2,p}(B_{2\delta}(x_0))$. Therefore, the regularity of u holds locally in a neighborhood of $x_0 \in U$. Since x_0 was chosen arbitrarily and since U is bounded, a standard covering argument yields the regularity of u up to the entire domain. That is, u belongs to $W^{2,p}(U)$. □

Remark 3.5. *In summary, a priori regularity estimates imply the actual regularity of weak solutions. From this point on, we study the regularity of solutions in various settings and function spaces; however, in most cases, we only establish the a priori estimates. It should be understood that the actual regularity of the solutions will follow from the a priori estimates as was done in this section.*

3.3 Bootstraping: Two Basic Examples

We show how to combine the previous $W^{2,p}$ a priori estimates with the Hölder estimates of Sections 1.4.3 and 1.4.4 (or more generally the Schauder estimates of Section 3.5 below) to get the smoothness of weak solutions to a simple linear PDE and a related semilinear problem. The goal here is to introduce and provide simple examples of bootstrap methods.

Let $n \geq 3$ and suppose $U \subset \mathbb{R}^n$ is a bounded open subset with C^1 boundary. Consider the linear problem

$$\begin{cases} -\Delta u = c(x)u & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{3.29}$$

and the semilinear problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{3.30}$$

We shall prove that if $u \in H_0^1(U)$ is a weak solution of either problem, then it is actually smooth and therefore a classical solution. The idea is to treat each PDE as a linear equation with an integrable coefficient, then we apply the Sobolev embedding recursively to boost the integrability of u and verify it is Hölder continuous. The Schauder estimates will then show u is of class $C^{2,\alpha}$. Similarly, applying the Schauder estimates successively will further imply that the solution is in fact smooth.

Remark 3.6. *This idea of starting with a solution residing in a lower regularity space and iterating the a priori estimates to show it actually belongs to a higher regularity space is an example of a bootstrap procedure. We shall revisit bootstrap arguments again in the subsequent sections.*

Theorem 3.11. *Suppose that $u \in H_0^1(U)$ is a weak solution of problem* (3.29) *and $c(x)$ belongs to $L^{\frac{n}{2}}(U)$. Then u is smooth, i.e., $u \in C^\infty$.*

Proof. First, we show $u \in C^\alpha(U)$ for some $\alpha \in (0,1)$. By the Sobolev inequality, u belongs to $L^{\frac{2n}{n-2}}(U)$. Thus, Hölder's inequality ensures the source term $c(x)u$ belongs to $L^{\frac{2n}{n+2}}(U)$, since

$$\|cu\|_{L^{\frac{2n}{n+2}}(U)} \leq \|c\|_{L^{\frac{n}{2}}(U)} \|u\|_{L^{\frac{2n}{n-2}}(U)}.$$

Then L^p regularity theory implies $u \in W^{2,s_0}(U)$ where $s_0 = 2n/(n+2)$. Again, the Sobolev embedding $W^{2,s}(U) \hookrightarrow L^{\frac{ns}{n-2s}}(U)$ implies that u belongs to $L^{s_1}(U)$ where $s_1 = ns_0/(n-2s_0)$. From this, Hölder's inequality implies $c(x)u$ now belongs to $L^{s_1}(U)$ since

$$1/s_1 = 2/n + (n-2s_1)/ns_1$$

and

$$\|cu\|_{L^{s_1}(U)} \leq \|c\|_{L^{\frac{n}{2}}(U)} \|u\|_{L^{s_1}(U)}.$$

Thus L^p regularity theory ensures u belongs to $W^{2,s_1}(U)$. If $2s_1 > n$, the Sobolev embedding (see Theorem A.21) implies that u belongs to $C^\alpha(U)$ for some $\alpha \in (0,1)$; if $2s_1 = n$ then $u \in L^q(U)$ for all $1 \le q < \infty$ and we again deduce u is Hölder continuous. Otherwise, if $2s_1 < n$ and thus invoke the L^p and Sobolev estimates once again to deduce that $u \in W^{2,s_1}(U) \hookrightarrow L^{s_2}(U)$, where $s_2 = ns_1/(n-2s_1) = ns_0/(n-4s_0)$. Therefore $u \in W^{2,s_2}(U)$, and if $2s_2 \ge n$, we get that u belongs to $C^\alpha(U)$ for some $\alpha \in (0,1)$ and we are done. Otherwise, we may repeat this argument successively to find a suitably large j in which $2s_j > n$ and u belongs to $W^{2,s_j}(U)$. Hence, Sobolev embedding ensures $u \in C^\alpha(U)$ for some $\alpha \in (0,1)$.

Finally, applying the Schauder estimates repeatedly, we further deduce that u is smooth. □

Corollary 3.1. *Suppose* $1 < p < (n+2)/(n-2)$. *If* $u \in H_0^1(U)$ *is a weak solution of problem* (3.30)*, then* u *is smooth.*

Proof. Set $c(x) = |u|^{p-1}$. Since u belongs to $H_0^1(U)$, the Sobolev inequality implies that $u \in L^s(U)$ for $1 \le s \le 2n/(n-2)$. From this, it is easy to check that $c(x)$ belongs to $L^{\frac{n}{2}}(U)$. Hence, the previous theorem applies to get that u is smooth. □

3.4 Regularity in the Sobolev Spaces H^k

In this section, we show the regularity of weak solutions to uniformly elliptic equations in $H^2(U)$ or $W^{2,2}(U)$. Under the appropriate conditions, we shall establish both interior and boundary a priori estimates for the weak solutions to conclude that they are indeed strong solutions. Then, we iterate these estimates under the right conditions to conclude that the weak solutions belong to higher order Sobolev spaces. In fact, we show weak solutions are actually classical solutions if the data of the elliptic problem are smooth. We assume throughout the section that $U \subset \mathbb{R}^n$ is a bounded, open set and we take $u \in H_0^1(U)$ to be a weak solution of

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

where as always L is uniformly elliptic and is in divergence form, i.e.,

$$Lu = -\sum_{i,j=1}^{n} D_j \left(a^{ij}(x) D_i u\right) + \sum_{j=1}^{n} b^i(x) D_i u + c(x)u.$$

Of course, the regularity of the coefficients a^{ij}, b^i and c and the source term f must be specified for each regularity result.

3.4.1 Interior regularity

Theorem 3.12 (Interior H^2-regularity)**.** *Assume*

$$a^{ij} \in C^1(U),\ b^i,\ c \in L^\infty(U) \quad \textit{for } i,j = 1,2,\ldots,n, \tag{3.31}$$

and $f \in L^2(U)$. Suppose further that $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f \ \textit{ in } U.$$

Then u belongs to $H^2_{loc}(U)$ and thus is a strong solution of this elliptic PDE, and for each open subset $V \subset\subset U$ there holds the estimate

$$\|u\|_{H^2(V)} \le C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}), \tag{3.32}$$

where the positive constant C depends only on V, U and the coefficients of the operator L.

Remark 3.7. *Note that this theorem is not assuming u satisfies the Dirichlet boundary condition on ∂U. Also, recall that u is said to be a strong solution of the elliptic PDE if it is twice weakly differentiable and satisfies the equation $Lu = f$, for a.e. x in U. Indeed, this follows simply from the fact that u belongs to $H^2_{loc}(U)$. More precisely, the definition of a weak solution and integration by parts indicates that*

$$(Lu, v) = B[u, v] = (f, v)$$

for all $v \in C_c^\infty(U)$. Thus, from Corollary A.2, this shows that $Lu - f = 0$ a.e. or that $Lu = f$ for a.e. $x \in U$.

Proof of Theorem 3.12. Fix $V \subset\subset U$, choose an open W such that $V \subset\subset W \subset\subset U$, and select a smooth cut-off function ζ such that $0 \le \zeta \le 1$, $\zeta \equiv 1$ in V and $\zeta \equiv 0$ in W^C.

Step 1: Since $u \in H^1(U)$ is a weak solution of $Lu = f$ in U, there holds

$$\sum_{i,j=1}^n \int_U a^{ij}(x) D_i u D_j v \, dx = \int_U F v \, dx \ \text{ for every } \ v \in H_0^1(U), \tag{3.33}$$

where

$$F := f - \sum_{i=1}^n b^i(x) D_i u - c(x) u.$$

Step 2: Let $|h| > 0$ be small, choose $k \in \{1, 2, \ldots, n\}$ and substitute

$$v = -D_k^{-h}(\zeta^2 D_k^h u)$$

into (3.33) where $D_k^h u$ is the difference quotient

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h} \ \ (h \in \mathbb{R}\backslash\{0\}).$$

For this particular test function v, we denote the resulting left-hand side (respectively, right-hand side) of (3.33) by A (respectively, B). After some tedious calculations and denoting $v^h(x) := v(x + he_k)$, we calculate

$$\begin{aligned} A = &\sum_{i,j=1}^{n} \int_U a^{ij,h}(x) D_k^h D_i u D_k^h D_j u \zeta^2 \, dx + \sum_{i,j=1}^{n} \int_U [a^{ij,h}(x) D_k^h D_i u D_k^h u (2\zeta) D_j \zeta \\ &+ (D_k^h a^{ij}(x)) D_i u D_k^h D_j u \zeta^2 + (D_k^h a^{ij}(x)) D_i u D_k^h u (2\zeta) D_j \zeta] \, dx \\ =: &\, A_1 + A_2. \end{aligned}$$

Indeed, the uniform ellipticity condition implies

$$A_1 \geq \theta \int_U \zeta^2 |D_k^h Du|^2 \, dx.$$

In addition, from (3.31) we get

$$|A_2| \leq C \int_U \zeta |D_k^h Du| |D_k^h u| + \zeta |D_k^h Du| |Du| + \zeta |D_k^h u| |Du| \, dx,$$

for some constant $C > 0$. Thus, Cauchy's inequality with ϵ (see Theorem A.1) implies the estimate

$$|A_2| \leq \epsilon \int_U \zeta^2 |D_k^h Du|^2 \, dx + \frac{C}{\epsilon} \int_W |D_k^h u|^2 + |Du|^2 \, dx.$$

Choosing $\epsilon = \theta/2$ and using the fact that

$$\int_W |D_k^h u|^2 \, dx \leq C \int_U |Du|^2 \, dx,$$

we arrive at

$$|A_2| \leq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 \, dx + C \int_U |Du|^2 \, dx.$$

This estimate and the estimate of A_1 imply

$$A \geq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 \, dx - C \int_U |Du|^2 \, dx. \tag{3.34}$$

Recalling the definition of F and our particular choice of the test function v, we get

$$\begin{aligned} |B| &\leq C \int_U (|f| + |Du| + |u|) |v| \, dx \\ &\leq \epsilon \int_U \zeta^2 |D_k^h Du|^2 \, dx + \frac{C}{\epsilon} \int_U f^2 + u^2 + |Du|^2 \, dx \end{aligned}$$

where we used Cauchy's inequality with ϵ (Theorem A.1) and the fact that

$$\int_U |v|^2 \, dx \leq C \int_U |Du|^2 + \zeta^2 |D_k^h Du|^2 \, dx.$$

Choosing $\epsilon = \theta/4$, we arrive at

$$|B| \le \frac{\theta}{4}\int_U \zeta^2 |D_k^h Du|^2\, dx + C\int_U f^2 + u^2 + |Du|^2\, dx \le C(\|f\|^2_{L^2(U)} + \|u\|^2_{H^1(U)}). \tag{3.35}$$

Recalling that $A = B$ and inserting the estimates (3.34) and (3.35), we deduce that

$$\int_V |D_k^h Du|^2\, dx \le \int_U \zeta^2 |D_k^h Du|^2\, dx \le C\int_U f^2 + u^2 + |Du|^2\, dx$$

for $k = 1, 2, \ldots, n$, and all sufficiently small $|h| \neq 0$. This implies that $Du \in H^1_{loc}(U; \mathbb{R}^n)$. Hence, we have that $u \in H^2_{loc}(U)$ with the estimate

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}). \tag{3.36}$$

Step 3: Notice that we are not quite done; namely, it remains to replace the H^1 norm of u instead with its L^2 norm in the estimate (3.36).

Indeed, since $V \subset\subset W \subset\subset U$, the procedure above can be used to establish the interior estimate

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}) \tag{3.37}$$

for an appropriate positive constant C depending on V, W, etc. Choosing a new smooth cut-off function $0 \le \zeta \le 1$ with $\zeta \equiv 1$ in W, $supp(\zeta) \subset U$ and setting $v = \zeta^2$ in identity (3.33), elementary calculations will lead to the estimate

$$\int_U \zeta^2 |Du|^2\, dx \le C\int_U f^2 + u^2\, dx.$$

Hence,

$$\|u\|_{H^1(W)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}),$$

and inserting this into (3.37) completes the proof of the theorem. □

3.4.2 Higher interior regularity

By assuming stronger smoothness of the coefficients in the elliptic equation, we may iterate the previous interior regularity theorem to get the higher regularity of weak solutions. Namely, there holds the following.

Theorem 3.13 (Higher interior regularity)**.** *Let m be a non-negative integer, and assume*

$$a^{ij},\, b^i,\, c \in C^{m+1}(U) \quad \text{for } i, j = 1, 2, \ldots, n,$$

and

$$f \in H^m(U).$$

Suppose further that $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f \text{ in } U.$$

Then

$$u \text{ belongs to } H^{m+2}_{loc}(U), \tag{3.38}$$

and for each open subset $V \subset\subset U$ there holds the estimate

$$\|u\|_{H^{m+2}(V)} \leq C(\|u\|_{L^2(U)} + \|f\|_{H^m(U)}), \tag{3.39}$$

where the positive constant C depends only on m, V, U and the coefficients of the elliptic operator L.

Proof. We proceed by induction. Clearly, the case $m = 0$ holds by Theorem 3.12.
Step 1: Assume that assertions (3.38) and (3.39) hold for an arbitrary integer $m \geq 2$ and all open sets U, coefficients a^{ij}, b^i, c, etc. Now suppose

$$a^{ij},\, b^i,\, c \in C^{m+2}(U), \tag{3.40}$$

and

$$f \in H^{m+1}(U), \tag{3.41}$$

and $u \in H^1(U)$ is a weak solution of $Lu = f$ in U.

So by the induction hypothesis, there holds $u \in H^{m+2}_{loc}(U)$ with the interior estimate

$$\|u\|_{H^{m+2}_{loc}(W)} \leq C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}) \tag{3.42}$$

for each $W \subset\subset U$ and an appropriate positive constant C, depending only on W, the coefficients of L, etc. Now fix $V \subset\subset W \subset\subset U$.

Step 2: Now let α be any multi-index with $|\alpha| = m + 1$, and choose any test function $v_1 \in C^\infty_c(W)$. Inserting $v := (-1)^{|\alpha|} D^\alpha v_1$ into the weak solution definition $B[u, v] = (f, v)_{L^2(U)}$, elementary calculations will lead to the identity

$$B[u_1, v_1] = (f_1, v_1)_{L^2(U)} \tag{3.43}$$

where

$$u_1 := D^\alpha u \in H^1(W) \tag{3.44}$$

and

$$f_1 := D^\alpha f - \sum_{\beta \leq \alpha, \beta \neq \alpha} \binom{\alpha}{\beta} \Big[- \sum_{i,j=1}^{n} D_j \Big(D^{\alpha-\beta} a^{ij}(x) D^\beta D_i u \Big) + \sum_{i=1}^{n} D^{\alpha-\beta} b^i(x) D^\beta D_i u + D^{\alpha-\beta} c(x) D^\beta u \Big]. \tag{3.45}$$

Since (3.43) holds for each $v_1 \in C_c^\infty(W)$, we see that u_1 is a weak solution of $Lu = f_1$ in W. So in view of (3.40)–(3.42) and (3.44), we have $f_1 \in L^2(U)$ with

$$\|f_1\|_{L^2(W)} \leq C(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)}).$$

Step 3: From Theorem 3.12, we conclude that u_1 belongs to $H^2(V)$ with the estimate

$$\|u_1\|_{H^2(V)} \leq C(\|f_1\|_{L^2(W)} + \|u_1\|_{L^2(W)}) \leq C(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)}).$$

Since this estimate holds for each multi-index α with $|\alpha| = m+1$ and $u_1 = D^\alpha u$, we deduce that $u \in H^{m+3}(V)$ and

$$\|u\|_{H^{m+3}(V)} \leq C(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)}).$$

This completes the induction step for the case $m+1$, and this finishes the proof of the theorem. □

In fact, provided that the data of the problem are smooth, we can apply Theorem 3.13 successively to deduce that the weak solutions are actually smooth.

Theorem 3.14 (Infinite differentiability in the interior)**.** *Assume*

$$a^{ij},\ b^i,\ c \in C^\infty(U) \quad \textit{for } i,j = 1,2,\ldots,n$$

and

$$f \in C^\infty(U).$$

Suppose further that $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f \ \textit{ in } U.$$

Then u belongs to $C^\infty(U)$.

Proof. According to Theorem 3.13, u belongs to $H^m_{loc}(U)$ for each integer $m = 1,2,\ldots$. So by the general Sobolev inqualities (see Theorem A.21), u belongs to $C^k(U)$ for $k = 1,2,\ldots$. This completes the proof. □

3.4.3 Global regularity

Next, we extend the earlier interior regularity estimates up to the boundary, but not surprisingly, additional smoothness up to the boundary ∂U on the data of the problem are needed.

Theorem 3.15 (Boundary H^2-regularity). *Assume*

$$a^{ij} \in C^1(\bar{U}),\ b^i,\ c \in L^\infty(U) \quad \textit{for } i,j = 1,2,\ldots,n, \tag{3.46}$$

$f \in L^2(U)$ and the boundary ∂U is C^2. Suppose that $u \in H_0^1(U)$ is a weak solution of the boundary-value problem

$$\begin{cases} Lu = f & \textit{in } U, \\ \ \ u = 0 & \textit{on } \partial U. \end{cases} \tag{3.47}$$

Then $u \in H^2(U)$, and there holds the estimate

$$\|u\|_{H^2(U)} \leq C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}), \tag{3.48}$$

where the positive constant C depends only on U and the coefficients of L.

Remark 3.8. *Note that we are now prescribing a Dirichlet boundary condition on the solution of* (3.47)*. This boundary condition, of course, should be understood in the trace sense. In addition, if u is the unique weak solution of the Dirichlet problem, then estimate* (3.48) *simplifies to*

$$\|u\|_{H^2(U)} \leq C\|f\|_{L^2(U)},$$

since Theorem 2.9 implies that $\|u\|_{L^2(U)} \leq \tilde{C}\|f\|_{L^2(U)}$ where $\tilde{C}$ depends only on U and the coefficients of L.

Proof of Theorem 3.15. We first prove the theorem for the special case when U is the half-ball

$$U = B_1(0) \cap \mathbb{R}^n_+.$$

Step 1: Set $V = B_{1/2}(0) \cap \mathbb{R}^n_+$ and select a smooth cut-off function ζ for which $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ in $B_{1/2}(0)$, and $\zeta \equiv 0$ in $B_1(0)^C$. In particular, $\zeta \equiv 1$ in V and vanishes near the curved part of ∂U. Since u is a weak solution of (3.47), we have that

$$B[u,v] = (f,v) \ \text{ for all } \ v \in H_0^1(U),$$

and so

$$\sum_{i,j=1}^n \int_U a^{ij}(x) D_i u D_j v \, dx = \int_U Fv \, dx, \tag{3.49}$$

where

$$F := f - \sum_{i=1}^n b^i(x) D_i u - c(x)u.$$

Step 2: Now let $h > 0$ be small, choose $k \in \{1,2,\ldots,n-1\}$ and write

$$v := -D_k^{-h}(\zeta^2 D_k^h u).$$

Note that

$$\begin{aligned} v(x) &= -\frac{1}{h}D_k^{-h}(\zeta^2(x)[u(x+he_k)-u(x)]) \\ &= \frac{1}{h^2}\Big(\zeta^2(x-he_k)[u(x)-u(x-he_k)] - \zeta^2(x)[u(x+he_k)-u(x)]\Big) \ \ (x\in U). \end{aligned}$$

Then, since $u=0$ along $\{x_n=0\}$ in the trace sense and $\zeta=0$ near the curved portion of ∂U, we get that $v\in H_0^1(U)$. Then, substituting this particular choice of v into (3.49), we may write the resulting expression as $A=B$ where

$$A := \sum_{i,j=1}^n \int_U a^{ij}(x)D_iuD_jv\,dx \tag{3.50}$$

and

$$B := \int_U Fv\,dx. \tag{3.51}$$

Step 3: We estimate the terms A and B, but the steps are similar to the steps found in the proof of Theorem 3.12 so we omit the details. Namely, there holds

$$A \geq \frac{\theta}{2}\int_U \zeta^2|D_k^hDu|^2\,dx - C\int_U |Du|^2\,dx \tag{3.52}$$

and

$$|B| \leq \frac{\theta}{4}\int_U \zeta^2|D_k^hDu|^2\,dx + C\int_U f^2+u^2+|Du|^2\,dx, \tag{3.53}$$

for an appropriate positive constant C. Inserting estimates (3.52) and (3.53) into the expression $A=B$, we deduce

$$\int_V |D_k^hDu|^2\,dx \leq C\int_U f^2+u^2|Du|^2\,dx$$

for $k=1,2,\dots,n-1$. Thus, this implies that

$$D_ku\in H^1(V) \ \text{ for } \ k=1,2,\dots,n-1$$

with the estimate

$$\sum_{k,\ell=1,k+\ell<2n}^n \|D_{\ell k}u\|_{L^2(V)} \leq C(\|u\|_{H^1(U)}+\|f\|_{L^2(U)}). \tag{3.54}$$

Step 4: Notice that estimate (3.54) is missing the last term $\|D_{nn}u\|_{L^2(U)}$. We now estimate this term.

In view of Theorem 3.12 and the definition of the elliptic operator L, u is a strong solution of $Lu=f$ in V. That is,

$$-\sum_{i,j=1}^n a^{ij}(x)D_iuD_ju + \sum_{i=1}^n \tilde{b}^i(x)D_iu + c(x)u = f \tag{3.55}$$

where $\tilde{b}^i(x) := b^i(x) - \sum_{j=1}^n D_j a^{ij}(x)$ for $i = 1, 2, \ldots, n$. From this we can solve for the last term $D_{nn}u$, i.e.,

$$a^{nn}(x)D_{nn}u = -\sum_{i,j=1,\, i+j<2n} a^{ij}(x)D_{ij}u + \sum_{i=1}^n \tilde{b}^i(x)D_i u + c(x) - f. \tag{3.56}$$

From the uniform ellipticity condition, $\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$ for all $x \in U$, $\xi \in \mathbb{R}^n$. Thus, if we take $\xi = e_n = (0, 0, \ldots, 0, 1)$ in the last estimate, we get

$$a^{nn}(x) \geq \theta > 0 \text{ in } U. \tag{3.57}$$

Hence, combining this and the assumptions (3.46) with identity (3.56) gives us

$$|D_{nn}u| \leq C\Big(\sum_{i,j=1,\, i+j<2n}^{n} |D_{ij}u| + |Du| + |u| + |f|\Big) \text{ in } U. \tag{3.58}$$

Therefore, applying estimate (3.54) to this, we arrive at the estimate

$$\|u\|_{H^2(V)} \leq C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}) \tag{3.59}$$

for some appropriate positive constant C.

Step 5: We drop the assumption that U is a half-ball. In general, we may choose any point $x^0 \in \partial U$ and since ∂U is C^2, we may assume, upon relabelling and reorienting the axes if necessary, that

$$U \cap B_r(x^0) = \{x \in B_r(x^0) \,|\, x_n > \gamma(x_1, x_2, \ldots, x_{n-1})\}$$

for some $r > 0$ and some C^2 function $\gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$. As indicated at the beginning of this chapter, we can change variables and write

$$y = \Phi(x) \text{ and } x = \Psi(y).$$

Step 6: Choose $s > 0$ so small that the half-ball $U_1 = B_s(0) \cap \{y_n > 0\}$ lies in $\Phi(U \cap B_r(x^0))$. Set

$$V_1 = B_{s/2}(0) \cap \{y_n > 0\} \tag{3.60}$$

and define

$$u_1(y) := u(\Psi(y)) \text{ for } y \in U_1.$$

Then it turns out that

$$(i)\ u_1 \in H^1(U_1), \qquad (ii)\ u_1 = 0 \text{ on } \partial U_1 \cap \{y_n = 0\} \tag{3.61}$$

where property (ii) should be understood in the trace sense. Then, after some elementary calculations, we can deduce that this u_1 is a weak solution of the PDE

$$L_1 u = f_1 \text{ in } U_1$$

where

$$f_1(y) = f(\Psi(y))$$

and

$$L_1 u = -\sum_{k,\ell=1}^{n} D_\ell(a_1^{k\ell} D_k u_1) + \sum_{k=1}^{n} b_1^k(x) D_k u + c_1(c)u$$

with

$$\begin{aligned}
a_1^{k\ell}(y) &= \sum_{r,s=1}^{n} a^{rs}(\Psi(y))\Phi^k_{x_r}(\Psi(y))\Phi^\ell_{x_s}(\Psi(y)) \quad (k,\ell = 1,2,\ldots,n),\\
b_1^k(y) &= \sum_{r=1}^{n} b^r(\Psi(y))\Phi^k x_r(\Psi(y)) \quad (k = 1,2,\ldots,n),\\
c_1(y) &= c(\Psi(y)).
\end{aligned}$$

Then, it turns out that L_1 is a uniformly elliptic operator and the matrix coefficient $a_1^{k\ell}(x)$ is C^1 since Φ and Ψ are C^2 maps.

Step 7: Applying our results from Steps 1–4 to the elliptic problem $L_1 u = f_1$ in U_1 and recalling (3.60), we deduce that $u_1 \in H^2(V_1)$ with the estimate

$$\|u_1\|_{H^2(V_1)} \leq C(\|u_1\|_{L^2(U_1)} + \|f_1\|_{L^2(U_1)}),$$

and so

$$\|u\|_{H^2(V)} \leq C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}) \tag{3.62}$$

for $V := \Psi(V_1)$.

Step 8: Finally, since ∂U is compact, we can cover it with finitely many sets $V_1, V_2, \ldots, V_N$ as above in which the estimate (3.62) holds in each V_i. Summing up these estimates over all V_i and combining the resulting estimate with the interior regularity estimate shows that $u \in H^2(U)$ with

$$\|u\|_{H^2(U)} \leq C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}).$$

This completes the proof of the theorem. □

3.4.4 Higher global regularity

Theorem 3.16 (Higher boundary regularity)**.** *Let m be a non-negative integer, and assume*

$$a^{ij}, b^i, c \in C^{m+1}(\bar{U}) \quad \text{for } i,j = 1,2,\ldots,n, \tag{3.63}$$

$$f \in H^m(U) \tag{3.64}$$

and the boundary ∂U is C^{m+2}. Suppose that $u \in H_0^1(U)$ is a weak solution of the boundary-value problem

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Then $u \in H^{m+2}(U)$, and there holds the estimate

$$\|u\|_{H^{m+2}(U)} \leq C(\|u\|_{L^2(U)} + \|f\|_{H^m(U)}), \tag{3.65}$$

where the positive constant C depends only on m, U and the coefficients of the elliptic operator L.

Proof. We only prove the boundary estimate for the special case when the domain is the half-ball $U = B_s(0) \cap \mathbb{R}^n_+$ for some $s > 0$. Proving it for a general domain U involves similar ideas as in the preceding theorem by straightening out the boundary and applying a standard covering argument.

Fix $t \in (0, s)$ and set $V = B_t(0) \cap \mathbb{R}^n_+$.

Step 1: We proceed by induction on the non-negative integer m with the goal of showing that (3.63) and (3.64), whenever $u = 0$ along $\{x_n = 0\}$ in the trace sense, imply $u \in H^{m+2}(V)$ with the estimate

$$\|u\|_{H^{m+2}(V)} \leq C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}),$$

for some positive constant C depending only on U, V and the coefficients of the operator L. Of course, the case $m = 0$ is a direct consequence of the preceding theorem.

Suppose then that

$$(i)\ a^{ij},\, b^i,\, c \in C^{m+2}(\bar{U}), \qquad (ii)\ f \in H^{m+1}(U), \tag{3.66}$$

u is a weak solution of

$$Lu = f \ \text{ in } \ U,$$

and u vanishes along $\{x_n = 0\}$ in the trace sense. Fix any $0 < t < r < s$ and write $W = B_r(0) \cap \mathbb{R}^n_+$. By the induction assumption, we have $u \in H^{m+2}(W)$ with

$$\|u\|_{H^{m+2}(W)} \leq C(\|u\|_{L^2(U)} + \|f\|_{H^m(U)}). \tag{3.67}$$

Furthermore, according to the interior regularity result of Theorem 3.13, $u \in H^{m+3}_{loc}(U)$.

Step 2: Let α be any multi-index with $|\alpha| = m + 1$ and $\alpha_n = 0$. Then set $u_1 := D^\alpha u$, which belongs to $H^1(U)$ and vanishes along the plane $\{x_n = 0\}$ in the trace sense. Furthermore, as in the proof of Theorem 3.13, u_1 is a weak solution of

$$L_1 u = f_1,$$

where

$$f_1 := D^\alpha f - \sum_{\beta\le\alpha,\,\beta\neq\alpha} \binom{\alpha}{\beta}\Big[\sum_{i,j=1}^{n} -\Big(D^{\alpha-\beta}a^{ij}(x)D^\beta D_i u\Big) + \sum_{i=1}^{n} D^{\alpha-\beta}b^i(x)D^\beta D_i u + D^{\alpha-\beta}c(x)D^\beta u\Big].$$

So in view of (3.63), (3.64), (3.66)(ii) and (3.67), we see that $f_1 \in L^2(W)$ with

$$\|f_1\|_{L^2(W)} \le C(\|u\|_{L^2(U)} + \|f\|_{H^{m+1}(U)}).$$

From our proof of Theorem 3.15, we can deduce that $u_1 \in H^2(V)$ with

$$\|u_1\|_{H^2(V)} \le C(\|u_1\|_{L^2(W)} + \|f_1\|_{L^2(W)}) \le C(\|u\|_{L^2(U)} + \|f\|_{H^{m+1}(U)}).$$

Noting that $u_1 = D^\alpha u$, this shows that

$$\|D^\beta u\|_{L^2(V)} \le C(\|u\|_{L^2(U)} + \|f\|_{H^{m+1}(U)})$$

for any multi-index β with $|\beta| = m+3$ and $\beta_n = 0, 1,$ or 2.

Step 3: We only need to remove the previous restriction on β_n, and we do so by induction. Namely, assume that

$$\|D^\beta u\|_{L^2(V)} \le C(\|u\|_{L^2(U)} + \|f\|_{H^{m+1}(U)})$$

for any multi-index β with $|\beta| = m+3$ and $\beta_n = 0, 1, 2, \ldots, j$ for some $j \in \{2, 3, \ldots, m+2\}$. Assume then $|\beta| = m+3$, $\beta_n = j+1$. Let us write $\beta = \gamma + \delta$ for $\delta = (0, \ldots, 0, 2)$ and $|\gamma| = m+1$. Since, $u \in H^{m+3}_{loc}(U)$ and $Lu = f$ in U, we have $D^\gamma Lu = D^\gamma f$ a.e. in U. Now, $D^\gamma Lu = a^{nn}(x)D^\beta u + T$ where T is a sum of terms involving at most j derivatives of u with respect to x_n and at most $m+3$ derivatives with respect to all the other variables. Since $a^{nn}(x) \ge \theta > 0$ in U, the initial induction hypothesis imply that

$$\|D^\beta u\|_{L^2(V)} \le C(\|u\|_{L^2(U)} + \|f\|_{H^{m+1}(U)})$$

provided that $|\beta| = m+3$ and $\beta_n = j+1$. So by induction, we have

$$\|u\|_{H^{m+3}(V)} \le C(\|u\|_{L^2(U)} + \|f\|_{H^{m+1}(U)}).$$

This completes the proof. □

We have a global smoothness property of weak solutions to the Dirichlet problem provided the data are globally smooth.

Theorem 3.17 (Infinite differentiability up to the boundary). *Assume*

$$a^{ij}, b^i, c \in C^\infty(\bar{U}) \quad \textit{for } i,j = 1,2,\ldots,n,$$

$f \in C^\infty(\bar{U})$ and the boundary ∂U is C^∞. Suppose that $u \in H_0^1(U)$ is a weak solution of the boundary-value problem

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Then $u \in C^\infty(\bar{U})$.

Proof. According to Theorem 3.16, we have $u \in H^m(U)$ for each integer $m = 1,2,\ldots$. Thus, Theorem A.21 implies that u belongs to $C^k(\bar{U})$ for each $k = 1,2,\ldots$. This completes the proof of the theorem. □

3.5 The Schauder Estimates and $C^{2,\alpha}$ Regularity

This section develops the Schauder theory for classical solutions. We mainly state and prove the interior Schauder estimates. We also state the global estimates but refer the readers to [7] for details. Recall We have already developed this theory for Poisson's equation in subsection 1.4.3-1.4.4 and extending it to general uniformly elliptic second-order equations will require a little more effort.

At the expense of being redundant, we review some relevant terminology for convenience, but we already introduced in the first chapter. Let $U \subseteq \mathbb{R}^n$, $x_0 \in U$ and $\alpha \in (0,1]$. We denote by $C^k(\bar{U}) = C^{k,0}(\bar{U})$ the Banach space of functions f which are k-times continuously differentiable on $\bar{U}$ equipped with the norm

$$\|f\|_{k;U} := \sum_{j=0}^{k} [f]_{j;U}, \tag{3.68}$$

where $[f]_{j;U} := \sup_U |D^j f(x)|$.

For Hölder continuity, we introduce the corresponding class of spaces often called Hölder spaces. We say a function f is **Hölder continuous with exponent α at x_0** if the quantity

$$[f]_{\alpha,x_0} := \sup_U \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha}$$

is finite. Furthermore, if $\alpha = 1$, then f is said to be Lipschitz continuous at x_0. We say f is **Hölder continuous with exponent α in U** if

$$[f]_{\alpha;U} := \sup_{x,y \in U, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite. For $\alpha \in (0,1]$, we introduce the additional semi-norms

$$
\begin{aligned}
[f]_{0,0;U} = [f]_{0;U} &:= \sup_U |f(x)|, \\
[f]_{0,\alpha;U} = [f]_{\alpha;U} &:= \sup_{x_0 \in U} [f]_{\alpha,x_0}, \\
[f]_{k,0;U} = [f]_{k;U} &:= \sum_{|\beta|=k} [D^\beta f]_{0;U}, \\
[f]_{k,\alpha;U} &:= \sum_{|\beta|=k} [D^\beta f]_{\alpha;U}.
\end{aligned}
$$

Definition 3.5. *We denote by $C^{k,\alpha}(\bar{U})$, where $0 < \alpha \leq 1$, the space consisting of functions $f \in C^k(\bar{U})$ satisfying $[f]_{k,\alpha;U} < \infty$. This space is indeed a Banach space equipped with the norm*

$$
\|f\|_{k,\alpha;U} := \|f\|_{k;U} + [f]_{k,\alpha;U}. \tag{3.69}
$$

Remark 3.9. *We make some assertions about our notation above. We sometimes drop the set U in the subscripts of the semi-norms. For our notation on the norms, we often interchange $\|\cdot\|$ with $|\cdot|$, and vice versa. Moreover, when $0 < \alpha < 1$, $C^{k,\alpha}(\bar{U})$ is commonly called a Hölder space.*

Some useful properties of Hölder spaces are as follows.

Lemma 3.5. *For $u, v \in C^\alpha(\overline{U})$, where $0 < \alpha \leq 1$, there holds*

$$
[uv]_\alpha \leq [u]_0[v]_\alpha + [u]_\alpha[v]_0 \leq |u|_\alpha |v|_\alpha.
$$

Definition 3.6. *A domain U is said to satisfy the cone property if there exists a finite cone V such that for any $x \in U$, there is a cone congruent to V with vertex x contained completely in U.*

Theorem 3.18. *Suppose that U satisfies the cone property with h the height of the cone. Then, for any $0 < \epsilon \leq h$, we have*

$$
[u]_2 \leq \epsilon^\alpha [u]_{2,\alpha} + \frac{C}{\epsilon^2}|u|_0, \tag{3.70}
$$

$$
[u]_1 \leq \epsilon^{1+\alpha} [u]_{2,\alpha} + \frac{C}{\epsilon}|u|_0, \tag{3.71}
$$

where the constant $C > 0$ depends only on n and solid angle of opening of the cone.

Let U be an bounded open domain and consider the general second-order linear elliptic equation

$$
-a^{ij}(x) D_{ij} u + b^i(x) D_i u + c(x) u = f \text{ in } U, \tag{3.72}
$$

where summations over the indices i and j are understood. As usual, we assume there exist $0 < \lambda \le \Lambda$ such that

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \text{ for all } x \in U,\, \xi \in \mathbb{R}^n, \tag{3.73}$$

$a^{ij}, b^i, c \in C^\alpha(\bar{U})$ $(0 < \alpha < 1)$ and

$$\frac{1}{\lambda}\Big\{\sum_{ij}\|a^{ij}\|_{\alpha;U} + \sum_i \|b^i\|_{\alpha;U} + \|c\|_{\alpha;U}\Big\} \le \Lambda_\alpha. \tag{3.74}$$

3.5.1 Some preliminary and intermediate results

We start with a gradient estimate for Poisson's equation.

Lemma 3.6. *Let $u \in C^\infty(\mathbb{R}^n)$ satisfy*

$$-\Delta u = f \quad \text{in } \mathbb{R}^n.$$

Then, for any $R > 0$, there holds

$$\max_{1\le i\le n} |D_i u(x)| \le \frac{n}{R} osc_{B_R(x)} u + R \sup_{B_R(x)} |f|, \tag{3.75}$$

where $osc_S u := \sup_{x\in S} u(x) - \inf_{x\in S} u(x)$ is called the oscillation of u on the set S.

Proof. Set $F_0 = \sup_{B_R(x)} |f|$, and we may assume $x = 0$ by translation invariance. An integration by parts shows

$$\begin{aligned}\int_{B_\rho(0)} \Delta(D_i u)\, dx &= \int_{\partial B_\rho(0)} \frac{\partial D_i u}{\partial \nu}\, dS = \rho^{n-1}\int_{\partial B_1(0)} \frac{\partial D_i u}{\partial \rho}(\rho\omega)\, dS_\omega \\ &= \rho^{n-1}\frac{\partial}{\partial\rho}\Big(\rho^{1-n}\int_{\partial B_\rho(0)} D_i u\, dS\Big).\end{aligned}$$

Alternatively, since $\Delta(D_i u) = D_i(\Delta u)$, integration by parts also yields

$$\int_{B_\rho(0)} \Delta(D_i u)\, dx = \int_{\partial B_\rho(0)} \Delta u \nu_i\, dS = -\int_{\partial B_\rho(0)} f\nu_i\, dS.$$

Combining these two identities lead us to

$$\pm\frac{\partial}{\partial\rho}\Big(\rho^{1-n}\int_{\partial B_\rho(0)} D_i u\, dS\Big) = \frac{1}{\rho^{n-1}}\Big|\int_{\partial B_\rho(0)} f\nu_i\, dS\Big| \le \omega_n F_0, \tag{3.76}$$

where recall that $\omega_n := |\mathbb{S}^{n-1}|$. Integrating (3.76) in $0 < \rho < r$ gives us

$$\pm\Big(r^{1-n}\int_{\partial B_r(0)} D_i u\, dS - \omega_n D_i u(0)\Big) \le \omega_n F_0 r.$$

Multiplying the last inequality by r^{n-1} and integrating in $0 < r < R$ and using polar coordinates (co-area formula), we deduce that

$$\pm\Big(\int_{B_R(0)} D_i u\, dx - \frac{\omega_n}{n} R^n D_i u(0)\Big) \leq \frac{\omega_n}{n+1} F_0 R^{n+1},$$

which implies

$$\begin{aligned}
|D_i u(0)| &\leq \frac{n}{n+1} F_0 R + \Big|\frac{n}{\omega_n R^n}\int_{B_R(0)} D_i u\, dx\Big| \\
&\leq F_0 R + \Big|\frac{n}{\omega_n R^n}\int_{B_R(0)} D_i(u - u(0))\, dx\Big| \\
&\leq F_0 R + \frac{n}{\omega_n R^n}\Big|\int_{\partial B_R(0)} (u(x) - u(0))\nu_i\, dS\Big| \\
&\leq F_0 R + \frac{n}{R} osc_{B_R(0)} u.
\end{aligned}$$

□

We slightly extend the $C^{2,\alpha}$ estimates for Poisson's equation and Laplace's operator to the elliptic problem

$$-a^{ij}(x) D_{ij} u = f, \tag{3.77}$$

where the coefficient matrix $A = (a^{ij})$ satisfies

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n, \tag{3.78}$$

for some $0 < \lambda \leq \Lambda$.

Theorem 3.19. *Let $0 < \alpha < 1$ and $u \in C_0^{2,\alpha}(\mathbb{R}^n)$ satisfy (3.77). Then following Hólder estimate holds,*

$$[D^2 u]_\alpha \leq \frac{C}{\lambda}[f]_\alpha,$$

where $C > 0$ depends only on α, n and Λ/λ.

Proof. We may assume $\lambda = 1$. After a suitable change of variables $y = Bx$ so that $B^T AB = Identity$, we can rewrite (3.77) into the form

$$-\Delta_y \bar{u} = \bar{f}.$$

Applying the $C^{2,\alpha}$ estimates from Theorem 1.32 gives us

$$[D^2\bar{u}]_\alpha \leq C[\bar{f}]_\alpha,$$

and thus this a priori estimate extends to u as well. □

3.5.2 The interior estimates

When establishing regularity a priori estimates, the following type of lemma is commonly applied with dilation arguments or when iterating inequalities. For instance, we will see it again when studying the De Giorgi–Nash–Moser regularity theory (see Section 3.7).

Lemma 3.7. *Let $\varphi(t)$ be a bounded non-negative function defined in the interval $[T_0, T_1]$, where $0 \le T_0 < T_1$. Suppose that φ satisfies*

$$\varphi(t) \le \theta\varphi(s) + \frac{A}{(s-t)^\alpha} + B \quad \text{for } T_0 \le t < s \le T_1, \tag{3.79}$$

where A, B and α are non-negative constants and $0 \le \theta < 1$. Then

$$\varphi(\rho) \le C\Big(\frac{A}{(R-\rho)^\alpha} + B\Big) \quad \text{for } T_0 \le \rho < R \le T_1, \tag{3.80}$$

where $C > 0$ depends only on α and θ.

Remark 3.10. *The idea is that if* (3.79) *holds, then we can improve it and essentially drop the term $\theta\varphi$.*

Proof. Set $t_0 = \rho$. For $i = 0, 1, 2, \ldots$, set

$$t_{i+1} = t_i + (1-\tau)\tau^i(R-\rho)$$

for some $0 < \tau < 1$ to be specified later. From (3.79), we get for $i = 0, 1, 2, \ldots$,

$$\varphi(t_i) \le \theta\varphi(t_{i+1}) + \frac{A}{[(1-\tau)\tau^i(R-\rho)]^\alpha} + B.$$

Iterating this, we get

$$\varphi(t_0) \le \theta^k\varphi(t_k) + \Big(\frac{A}{(1-\tau)^\alpha(R-\rho)^\alpha} + B\Big)\sum_{i=0}^{k-1}\theta^i\tau^{-i\alpha}.$$

Choosing τ so that $\theta/\tau^\alpha < 1$, sending $k \longrightarrow \infty$ reveals (3.80). □

We are ready to prove a localized interior Schauder estimate for the solutions to (3.72). Similar to obtaining the $W^{2,p}$ estimates earlier, we will use the method of frozen coefficients to treat (3.72) like the constant coefficient problem (3.77).

Lemma 3.8. *Consider the problem* (3.72)*, and suppose the conditions of* (3.73)-(3.74) *hold. Then there exists $R_0 \le 1$, depending only on n, α, Λ/λ and Λ_α, such that for any $0 < R \le R_0$ with $B_R(0) \subset U$ and any solution $u \in C_0^{2,\alpha}(B_R(0))$ of* (3.72)*, there holds*

$$[D^2u]_{\alpha;B_R(0)} \le C\Big\{\lambda^{-1}[f]_{\alpha;B_R(0)} + R^{-(2+\alpha)}|u|_{\alpha;B_R(0)}\Big\},$$

where the constant $C > 0$ depends only on n, α, Λ/λ and Λ_α.

Proof. We may assume that $\lambda = 1$. Fixing $x_0 \in U$, and by the method of freezing coefficients, we may rewrite (3.72) as

$$-a^{ij}(x)D_{ij}u = F,$$

where

$$F = f + (a^{ij}(x) - a^{ij}(x_0))D_{ij}u - b^i D_i u - cu.$$

Applying Theorem 3.19 to the above linear constant-coefficient problem leads to

$$[D^2u]_{\alpha;B_R(x_0)} \leq C[F]_{\alpha;B_R(x_0)} \leq C\Big\{[f]_{\alpha;B_R(x_0)} + R^\alpha[D^2u]_{\alpha;B_R(x_0)} + |u|_{2;B_R(x_0)}\Big\},$$

By the interpolation inequalities of (3.70) and (3.71) with $\epsilon = R/2$,

$$[D^2u]_{\alpha;B_R(x_0)} \leq CR^\alpha[D^2u]_{\alpha;B_R(x_0)} + C\Big\{[f]_{\alpha;B_R(x_0)} + R^{-2}|u|_{0;B_R(x_0)}\Big\}.$$

Letting $R_0 = (1/2C)^{1/\alpha}$, then for $0 < R \leq R_0$,

$$[D^2u]_{\alpha;B_R(x_0)} \leq C\Big\{[f]_{\alpha;B_R(x_0)} + R^{-2}|u|_{0;B_R(x_0)}\Big\}.$$

□

Theorem 3.20 (Interior Schauder estimates)**.** *For $\alpha \in (0,1)$, let $u \in C^{2,\alpha}(U)$ be a solution of (3.72). Then for $U' \subset\subset U$, we have*

$$\|u\|_{2,\alpha;U'} \leq C\left(\frac{1}{\lambda}\|f\|_{\alpha;U} + \|u\|_{0;U}\right),$$

where C depends only on $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ and $dist(U', \partial U)$.

3.5.3 The boundary and global estimates

Following similar ideas used in obtaining the interior estimates, we can establish corresponding boundary Schauder estimates.

Theorem 3.21 (Global Schauder estimates)**.** *Consider the same assumptions from the previous theorem and further assume $\partial U \in C^{2,\alpha}$. Suppose that $u \in C^{2,\alpha}(\bar{U})$ is a solution of (3.72) satisfying the boundary condition $u = g$ on ∂U where $g \in C^{2,\alpha}(\bar{U})$. Then*

$$\|u\|_{2,\alpha;U} \leq C\left(\frac{1}{\lambda}\|f\|_{\alpha;U} + \|g\|_{2,\alpha;U} + \|u\|_{0;U}\right),$$

where C depends only on $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ and U. Moreover, if u satisfies the maximum principle, then the last term on the right-hand side of the global estimate can be dropped.

3.6 Hölder Continuity for Weak Solutions: A Perturbation Approach

In this section, we prove the classical Hölder estimates for second-order elliptic equations using a perturbation approach. For the sake of simplicity, we consider the Dirichlet boundary value problem

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{3.81}$$

where

$$Lu = -\sum_{i,j=1}^{n} D_j \left(a^{ij}(x) D_i u \right) + c(x)u.$$

Recall that $u \in H_0^1(U)$ is a weak solution of (3.81) if

$$\int_U a^{ij}(x) D_i u D_j \varphi + c(x) u \varphi \, dx = \int_U f(x) \varphi \, dx \;\text{ for all }\; \varphi \in H_0^1(U).$$

As before, unless stated otherwise, we assume L is uniformly elliptic, $a^{ij} \in L^\infty(U)$, the coefficient $c \in L^{\frac{n}{2}}(U)$, and $f \in L^{\frac{2n}{n+2}}(U)$. Note that the assumptions on c and f and the Sobolev embedding allows for the weak solution definition to make sense. Now, a proper space to study the Hölder regularity properties in this perturbation framework are the Morrey and Campanato spaces.

3.6.1 Morrey–Campanato Spaces

Here, we shall provide the definitions and basic properties of certain subspaces of L^p spaces—the Morrey and Campanato spaces. These function spaces allow us to generalize the Sobolev inequalities and provide the proper setting for studying the Hölder regularity of weak solutions to elliptic equations. As usual, we let $U \subset \mathbb{R}^n$ be open (not necessarily bounded) and let $U_r(x) := B_r(x) \cap U$.

Definition 3.7 (Morrey Space). *Let $1 \leq p < \infty$ and $\lambda \geq 0$. The Morrey space $M^{p,\lambda}(U)$ is defined as*

$$M^{p,\lambda}(U) := \left\{ f \in L^p(U) \,\middle|\, \int_{U_r(x_0)} |f|^p \, dx \leq C^p \cdot r^\lambda \;\; \textit{for any} \;\; x_0 \in U, \, r > 0 \right\}$$

with norm

$$\|f\|_{M^{p,\lambda}(U)} := \left(\sup_{x_0 \in U, r > 0} \frac{1}{r^\lambda} \int_{U_r(x_0)} |f|^p \, dx \right)^{1/p}.$$

Proposition 3.5. *Let $1 \leq p < \infty$ and $\lambda \geq 0$. Then*

(i) $M^{p,\lambda}(U)$ is a Banach space,

(ii) $M^{p,0}(U) = L^p(U)$,

(iii) $M^{p,n}(U) = L^\infty(U)$,

(iv) *If* $q > p$ *then* $L^q(U) \hookrightarrow M^{p,\lambda}(U)$ *for* $\lambda = \lambda(p,q)$.

Definition 3.8 (Type A domains)**.** *A domain U is of type A if there exists a constant $A > 0$ such that for any $x_0 \in U$ and $0 < r < diam(U)$, $|U_r(x_0)| \geq A \cdot r^n$.*

Definition 3.9 (Campanato Space)**.** *Let $1 \leq p < \infty$ and $\lambda \geq 0$. The Campanato space $L^{p,\lambda}(U)$ is defined as*

$$L^{p,\lambda}(U) := \left\{ f \in L^p(U) \,\middle|\, [f]_{L^{p,\lambda}(U)} < \infty \right\}$$

where the Campanato seminorm is given by

$$[f]_{L^{p,\lambda}(U)} := \left(\sup_{x_0 \in U, r>0} \frac{1}{r^\lambda} \int_{U_r(x_0)} |f - (f)_{x_0,r}|^p \, dx \right)^{1/p}.$$

Remark 3.11. *Indeed, the quantity $[f]_{L^{p,\lambda}(U)}$ is a seminorm as any constant function f satisfies $[f]_{L^{p,\lambda}(U)} = 0$.*

Proposition 3.6. *Let $1 \leq p < \infty$ and $\lambda \geq 0$. Then*

(i) If U is of type A and $0 < \lambda < n$, then $M^{p,\lambda}(U) = L^{p,\lambda}(U)$,

(ii) If $\lambda = n$ and $p = 1$, then $L^{1,n}(U) = BMO(U)$ for any U,

(iii) If $\lambda > n + p$, then for any U and any p, $L^{p,\lambda}(U)$ is trivial in that it only contains constant functions.

Remark 3.12. *To summarize, the Morrey and Campanato spaces are indistinguishable in the range $\lambda \in (0, n)$. In the endpoint case $p = 1$ and $\lambda = n$, the Campanato space reduces to the space BMO, which is larger and properly contains the space $L^\infty(U) = M^{p,n}(U)$. In the interval $\lambda \in (n, n+p]$ we shall see that the Campanato spaces are indistinguishable from the Hölder and Lipschitz spaces, and this is precisely the setting for studying the Hölder regularity of weak solutions to elliptic equations. Of course, when $\lambda > n+p$, the Campanato spaces (just as with the $C^\alpha(U)$ spaces when $\alpha > 1$) are trivial consisting of only the constant functions.*

We start with the following important embedding property.

Theorem 3.22 (Sobolev–Morrey Embedding)**.** *Let $U \subset \mathbb{R}^n$ be of type A, $1 \leq p < \infty$ and $\alpha \in (0, 1)$. If $u \in W^{1,p}(U)$ such that $Du \in L^{p,n-p+p\alpha}(U)$, then $u \in C^\alpha(U)$.*

Notice that this is a generalization of Morrey's inequality and Theorem A.20; that is, we recover Theorem A.20 from this if $p > n$ and $\alpha = 1 - n/p$ (or $n - p + p\alpha = 0$). Now to prove Theorem 3.22, we will need the next result, which indicates that the Campanato space $L^{p,\lambda}(U)$ is equivalent to the Hölder space $C^\alpha(U)$ for $1 \leq p < \infty$ and $\lambda = n + p\alpha$ with $\alpha \in (0,1)$. Indeed, this illustrates an important application of the Morrey–Campanato spaces when studying the Hölder regularity of weak solutions to elliptic equations.

Theorem 3.23. *Suppose the domain $U \subset \mathbb{R}^n$ is of type A and let $\alpha \in (0,1)$, then*

$$L^{p,n+p\alpha}(U) = C^\alpha(U).$$

Proof. First, we prove that $C^\alpha(U) \hookrightarrow L^{p,n+p\alpha}(U)$. Observe that

$$|f(x) - (f)_{x,r}| \leq \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(x) - f(y)|\, dy \leq Cr^\alpha [f]_{C^\alpha(U)}.$$

Thus,

$$\begin{aligned}
\frac{1}{r^{n+p\alpha}}\int_{B_r(x_0)} |f(x) - (f)_{x_0,r}|^p\, dx &\leq \frac{1}{r^{n+p\alpha}}\int_{B_r(x_0)} |f(x) - f(x_0) + f(x_0) - (f)_{x_0,r}|^p\, dx \\
&\leq \frac{C}{r^{n+p\alpha}}\int_{B_r(x_0)} \Big(\frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha} + [f]_{C^\alpha(U)}\Big)^p |x - x_0|^{p\alpha}\, dx \\
&\leq \frac{C[f]^p_{C^\alpha(U)}}{r^{n+p\alpha}}\int_{B_r(x_0)} |x - x_0|^{p\alpha}\, dx \\
&\leq C[f]^p_{C^\alpha(U)} r^{-n-p\alpha}\int_0^r t^{n+p\alpha}\,\frac{dt}{t} \\
&\leq C[f]^p_{C^\alpha(U)}.
\end{aligned}$$

This implies that

$$\|f\|_{L^{p,n+p\alpha}(U)} \leq C\|f\|_{C^\alpha(U)},$$

and so $f \in L^{p,n+p\alpha}(U)$ whenever $f \in C^\alpha(U)$, i.e., $C^\alpha(U) \hookrightarrow L^{p,n+p\alpha}(U)$. Hence, it only remains to prove that $L^{p,n+p\alpha}(U) \hookrightarrow C^\alpha(U)$. For simplicity we only give the proof of this for the case $p = 2$ (see Theorem 3.24 below), since our Hölder regularity results only considers weak solutions belonging to $H^1(U) = W^{1,p=2}(U)$. □

Proof of Theorem 3.22. This clearly follows from Theorem 3.23 and Poincaré's inequality. □

3.6.2 Preliminary Estimates

The following basically states and proves special cases of Theorems 3.22 and 3.23.

Theorem 3.24. *Suppose $u \in L^2(U)$ satisfies*

$$\int_{B_r(x)} |u - u_{x,r}|^2 \, dx \le M^2 r^{n+2\alpha} \;\; \textit{for any} \;\; B_r(x) \subset U$$

for some $\alpha \in (0,1)$. Then $u \in C^\alpha(U)$ and for any $U' \subset\subset U$ there holds

$$\|u\|_{C^\alpha(U')} \le C(M + \|u\|_{L^2}),$$

where $C = C(n, \alpha, U', U)$ and $\|u\|_{C^\alpha(U')} := \sup_{U'} |u| + \sup_{x,y \in U', x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha}$.

Proof. **Uniform Estimate:** Denote $R_0 = dist(U', \partial U)$. For any $x_0 \in U'$ and $0 < r_1 < r_2 \le R_0$, we have

$$\begin{aligned} |u_{x_0,r_1} - u_{x_0,r_2}|^2 &\le (|u(x) - u_{x_0,r_1}| + |u(x) - u_{x_0,r_2}|)^2 \\ &\le |u(x) - u_{x_0,r_1}|^2 + 2|u(x) - u_{x_0,r_1}||u(x) - u_{x_0,r_2}| + |u(x) - u_{x_0,r_2}|^2 \\ &\le 2(|u(x) - u_{x_0,r_1}|^2 + |u(x) - u_{x_0,r_1}|^2), \end{aligned}$$

where we applied Young's inequality: $2ab \le a^2 + b^2$ for $a, b \in \mathbb{R}$. Integrating this with respect to x in $B_{r_1}(x_0)$ yields

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \cdot \frac{\omega_n}{n} r_1^n = 2\Big\{ \int_{B_{r_1}(x_0)} |u - u_{x_0,r_1}|^2 \, dx + \int_{B_{r_2}(x_0)} |u - u_{x_0,r_2}|^2 \, dx \Big\},$$

from which the estimate

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \le C(n) M^2 r_1^{-n} \left(r_1^{n+2\alpha} + r_2^{n+2\alpha} \right) \tag{3.82}$$

follows. For any $R \le R_0$, with $r_1 = R/2^{i+1}$, $r_2 = R/2^i$, we obtain

$$|u_{x_0, 2^{-(i+1)}R} - u_{x_0, 2^{-i}R}| \le C(n) 2^{-(i+1)\alpha} M R^\alpha.$$

Thus, for any $h < k$,

$$|u_{x_0, 2^{-h}R} - u_{x_0, 2^{-k}R}| \le \frac{C(n)}{2^{(h+1)\alpha}} M R^\alpha \sum_{i=0}^{k-h-1} 2^{-i\alpha} \le \frac{C(n,\alpha)}{2^{h\alpha}} M R^\alpha.$$

This shows that $\{u_{x_0, 2^{-i}R}\} \subset \mathbb{R}$ is a Cauchy sequence, and therefore convergent whose limit $\bar{u}(x_0)$ is independent of the choice of R, since (3.82) can be applied with $r_1 = 2^{-i}R$ and $r_2 = 2^{-i}\bar{R}$ whenever $0 < R < \bar{R} \le R_0$. Thus, we obtain

$$\bar{u}(x_0) = \lim_{r \longrightarrow 0} u_{x_0,r} \;\; \text{and} \;\; |u_{x_0,r} - \bar{u}(x_0)| \le C(n) M r^\alpha \tag{3.83}$$

for any $0 < r \le R_0$. Recall that by Lebesgue's differentiation theorem, $\{u_{x,r}\}$ converges to u in $L^1(U)$ as $r \longrightarrow 0$, so we have $u = \bar{u}$ a.e. and the inequality in (3.83) implies $\{u_{x,r}\}$

converges uniformly to $u(x)$ in U'. Moreover, since $x \mapsto u_{x,r}$ is continuous for any $r > 0$, $u(x)$ is continuous. Again, by the estimate in (3.83), we get

$$|u(x)| \le CMR^\alpha + |u_{x,R}|$$

for any $x \in U'$ and $R \le R_0$. Hence, u is bounded in U' where

$$\|u\|_{L^\infty(U')} \le C\left(MR_0^\alpha + \|u\|_{L^2(U)}\right).$$

Hölder Estimate: Let $x, y \in U'$ with $R = |x-y| < R_0/2$. Then we have

$$|u(x) - u(y)| \le |u(x) - u_{x,2R}| + |u(y) - u_{y,2R}| + |u_{x,2R} - u_{y,2R}|.$$

The first two terms are estimated by the inequality in (3.83). For the last term, we rewrite it

$$|u_{x,2R} - u_{y,2R}| \le |u_{x,2R} - u(\zeta)| + |u_{y,2R} - u(\zeta)|,$$

and integrating with respect to ζ over $B_{2R}(x) \cap B_{2R}(y)$, which contains $B_R(x)$, yields

$$\begin{aligned} |u_{x,2R} - u_{y,2R}|^2 &\le \frac{2}{|B_R(x)|}\Big\{\int_{B_{2R}(x)} |u - u_{x,2R}|^2\,dx + \int_{B_{2R}(y)} |u - u_{y,2R}|^2\,dx\Big\} \\ &\le C(n,\alpha)M^2R^{2\alpha}. \end{aligned}$$

Hence,

$$|u(x) - u(y)| \le C(n,\alpha)M|x-y|^\alpha.$$

For $|x-y| > R_0/2$ we obtain

$$|u(x) - u(y)| \le 2\|u\|_{L^\infty(U')} \le C\Big\{M + \frac{1}{R_0^\alpha}\|u\|_{L^2}\Big\}|x-y|^\alpha.$$

This completes the proof. □

As remarked earlier, a consequence of this result is a special case of Theorem 3.22.

Corollary 3.2. *Suppose $u \in H^1_{loc}(U)$ satisfies*

$$\int_{B_r(x)} |Du|^2\,dx \le M^2 r^{n-2+2\alpha} \text{ for any } B_r(x) \subset U$$

for some $\alpha \in (0,1)$. Then $u \in C^\alpha(U)$ and for any $U' \subset\subset U$ there holds

$$\|u\|_{C^\alpha(U')} \le C(M + \|u\|_{L^2}),$$

where $C = C(n, \alpha, U', U)$

Proof. From Poincaré's inequality, we have

$$\int_{B_r(x)} |u - u_{x,r}|^2\,dx \le C(n)r^2 \int_{B_r(x)} |Du|^2\,dx \le C_n M^2 r^{n+2\alpha},$$

and the result follows immediately from the previous theorem. □

3.6.3 Hölder Continuity of Weak Solutions

First, we state two lemmas, which are key to establishing the Hölder continuity of weak solutions. The estimates in the resulting regularity theorems in this section are sometimes called Cordes-Nirenberg type estimates.

Lemma 3.9. *Let $\varphi(t)$ be a non-negative and non-decreasing function on $[0, R]$. Suppose that*

$$\varphi(\rho) \leq A\Big\{ \Big(\frac{\rho}{r}\Big)^{\alpha} + \epsilon\Big\}\varphi(r) + Br^{\beta} \ \textit{ for any } \ 0 < \rho \leq r \leq R, \tag{3.84}$$

where A, B, α, β are non-negative constants and $\beta < \alpha$. Then, for any $\gamma \in (\beta, \alpha)$, there exists a constant $\epsilon_0 = \epsilon_0(A, \alpha, \beta, \gamma)$ such that if $\epsilon < \epsilon_0$, we have for all $0 < \rho \leq r \leq R$

$$\varphi(\rho) \leq C\Big\{ \Big(\frac{\rho}{r}\Big)^{\gamma} \varphi(r) + B\rho^{\beta}\Big\},$$

where $C = C(A, \alpha, \beta, \gamma) > 0$. In particular, we have for any $0 < r \leq R$,

$$\varphi(r) \leq C\Big\{ \frac{\varphi(R)}{R^{\gamma}} r^{\gamma} + Br^{\beta}\Big\}.$$

Proof. For $\tau \in (0, 1)$ and $r \in (0, R)$, we rewrite (3.84) as

$$\varphi(\tau r) \leq \tau^{\gamma}(1 + \epsilon\tau^{-\alpha})\varphi(r) + Br^{\beta}.$$

Choosing τ so that $2A\tau^{\alpha} = \tau^{\gamma}$ and assuming $\epsilon_0\tau^{-\alpha} < 1$, we get

$$\varphi(\tau r) \leq \tau^{\gamma}\varphi(r) + Br^{\beta} \text{ for each } r < R.$$

Iterating this for all positive integers k, we obtain

$$\begin{aligned}\varphi(\tau^{k+1} r) &\leq \tau^{\gamma}\varphi(\tau^{k} r) + B\tau^{k\beta}r^{\beta} \leq \tau^{(k+1)\gamma}\varphi(r) + B\tau^{k\beta}r^{\beta}\sum_{j=0}^{k}\tau^{j(\gamma-\beta)} \\ &\leq \tau^{(k+1)\gamma}\varphi(r) + \frac{B\tau^{k\beta}r^{\beta}}{1 - \tau^{\gamma-\beta}}.\end{aligned}$$

From this, we choose k so that $\tau^{k+2}r < \rho \leq \tau^{k+1}r$ and we arrive at

$$\varphi(\rho) \leq \frac{1}{\tau^{\gamma}}\Big(\frac{\rho}{r}\Big)^{\gamma}\varphi(r) + \frac{B\rho^{\beta}}{\tau^{2\beta}(1 - \tau^{\gamma-\beta})}.$$

□

Lemma 3.10. *Suppose $u \in H^1(U)$ satisfies*

$$\int_{B_r(x_0)} |Du|^2 \, dx \leq Mr^{\mu} \ \textit{ for any } \ B_r(x_0) \subset U,$$

for some $\mu \in [0, n)$*. Then for any* $U' \subset\subset U$ *there holds for any* $B_r(x_0) \subset U$ *with* $x_0 \in U'$

$$\int_{B_r(x_0)} |u|^2\,dx \le C(n,\lambda,\mu,U,U')\left(M + \int_U |u|^2\,dx\right) r^\lambda,$$

where $\lambda = \mu + 2$ *if* $\mu < n-2$ *and* $\lambda \in [0,n)$ *if* $n - 2 \le \mu < n$*.*

Proof. From Poincaré's inequality,

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2\,dx \le Cr^2 \int_{B_r(x_0)} |Du|^2\,dx \le c(n)Mr^{\mu+2}$$

for any $x_0 \in U'$ and $0 < r \le R_0 := dist(U', \partial U)$. Hence,

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2\,dx \le c(n)Mr^\lambda$$

where λ is as stated in the lemma. Then for any $0 < \rho < r \le R_0$, we have

$$\begin{aligned}\int_{B_\rho(x_0)} u^2\,dx &\le 2\int_{B_\rho(x_0)} |u_{x_0,r}|^2\,dx + 2\int_{B_\rho(x_0)} |u - u_{x_0,r}|^2\,dx \\ &\le c(n)\rho^n |u_{x_0,r}|^2 + 2\int_{B_r(x_0)} |u - u_{x_0,r}|^2\,dx \\ &\le c(n)\Big(\frac{\rho}{r}\Big)^n \int_{B_r(x_0)} u^2\,dx + Mr^\lambda,\end{aligned}$$

where we used $|u_{x_0,r}|^2 \le \frac{c(n)}{r^n}\int_{B_r(x_0)} u^2\,dx$. Indeed, it follows that $\varphi(r) = \int_{B_r(x_0)} u^2\,dx$ satisfies

$$\varphi(\rho) \le c(n)\Big[\Big(\frac{\rho}{r}\Big)\varphi(r) + Mr^\lambda\Big] \text{ for any } 0 < \rho < r \le R_0.$$

Therefore, Lemma 3.9 implies that for any $0 < \rho < r \le R_0$,

$$\int_{B_\rho(x_0)} u^2\,dx \le c\Big[\Big(\frac{\rho}{r}\Big)^\lambda \int_{B_r(x_0)} u^2\,dx + M\rho^\lambda\Big].$$

In particular, if $r = R_0$,

$$\int_{B_\rho(x_0)} u^2\,dx \le c\rho^\lambda\Big(M + \int_U u^2\,dx\Big) \text{ for } 0 < \rho \le R_0.$$

□

For simplicity, we assume that $U = B_1 = B_1(0)$.

Theorem 3.25. *Let $u \in H^1(B_1)$ be a weak solution of* (3.81)*. Assume $a^{ij} \in C(\bar{B}_1)$, $c \in L^n(B_1)$, and $f \in L^q(B_1)$ for some $q \in (n/2, n)$. Then $u \in C^\alpha(B_1)$ where $\alpha = 2 - n/q \in (0,1)$. Moreover, there exists an $R_0 = R_0(\lambda, \Lambda, \tau, \|c\|_{L^n})$ such that for any $x_0 \in B_{1/2}$ and $r \le R_0$, there holds*

$$\int_{B_r(x_0)} |Du|^2 \, dx \le C r^{n-2+2\alpha} \Big\{ \|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)} \Big\},$$

where $C = C(\lambda, \Lambda, \tau, \|c\|_{L^n})$ is a positive constant with

$$|a^{ij}(x) - a^{ij}(y)| \le \tau |x - y| \quad \textit{for any } \; x, y \in B_1.$$

Remark 3.13. *In the case where $c \equiv 0$, we may replace $\|u\|_{H^1(B_1)}$ with $\|Du\|_{L^2(B_1)}$.*

The main idea in the proof is to compare the solution with harmonic functions and use a perturbation argument. So we rely on the previous estimates and comparison results on harmonic functions.

Lemma 3.11 (Basic Estimates for Harmonic Functions)**.** *Suppose $\{a^{ij}\}$ is a constant positive definite matrix satisfying the uniformly elliptic condition,*

$$\lambda |\xi|^2 \le a^{ij} \xi_i \xi_j \le \Lambda |\xi|^2 \quad \textit{for any } \; \xi \in \mathbb{R}^n$$

for some $0 < \lambda \le \Lambda$. Suppose $w \in H^1(B_r(x_0))$ is a weak solution of $D_i(a^{ij}(x) D_j w) = 0$ in $B_r(x_0)$. Then for any $0 < \rho \le r$, there hold

$$\int_{B_\rho(x_0)} |Dw|^2 \, dx \le C \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Dw|^2 \, dx,$$

$$\int_{B_\rho(x_0)} |Dw - (Dw)_{x_0,\rho}|^2 \, dx \le C \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Dw - (Dw)_{x_0,r}|^2 \, dx,$$

where $C = C(\lambda, \Lambda)$.

Proof. This follows from Lemma 1.3 with u replaced by Dw instead. $\square$

Lemma 3.12 (Comparison with Harmonic Functions)**.** *Suppose w is as in the previous lemma. Then for any $u \in H^1_0(B_r(x_0))$ there hold for any $0 < \rho \le r$,*

$$\int_{B_\rho(x_0)} |Du|^2 \, dx \le C \Big\{ \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Du|^2 \, dx + \int_{B_r(x_0)} |D(u - w)|^2 \, dx \Big\},$$

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,r}|^2 \, dx \le C \Big\{ \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 \, dx + \int_{B_r(x_0)} |D(u - w)|^2 \, dx \Big\},$$

where $C = C(\lambda, \Lambda)$.

Proof. We prove this by directly by simple computations. With $v = u - w$ we have that for any $0 < \rho \le r$,

$$\begin{aligned}
\int_{B_\rho(x_0)} |Du|^2\,dx &\le 2\int_{B_\rho(x_0)} |Dw|^2\,dx + 2\int_{B_\rho(x_0)} |Dv|^2\,dx \\
&\le C\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Dw|^2\,dx + 2\int_{B_r(x_0)} |Dv|^2\,dx \\
&\le C\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Du|^2\,dx + C\Big\{1 + \left(\frac{\rho}{r}\right)^n\Big\}\int_{B_r(x_0)} |Dv|^2\,dx,
\end{aligned}$$

and

$$\begin{aligned}
\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2\,dx &\le 2\int_{B_\rho(x_0)} |Du - (Dw)_{x_0,\rho}|^2\,dx + 2\int_{B_\rho(x_0)} |Dv|^2\,dx \\
&\le 4\int_{B_\rho(x_0)} |Dw - (Dw)_{x_0,\rho}|^2\,dx + 6\int_{B_\rho(x_0)} |Dv|^2\,dx \\
&\le C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Dw - (Dw)_{x_0,\rho}|^2\,dx + 6\int_{B_r(x_0)} |Dv|^2\,dx \\
&\le C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,\rho}|^2\,dx \\
&\qquad + C\Big\{1 + \left(\frac{\rho}{r}\right)^{n+2}\Big\}\int_{B_r(x_0)} |Dv|^2\,dx.
\end{aligned}$$

□

Proof of Theorem 3.25. We decompose u into a sum $v + w$ where w satisfies a homogeneous equation and v has estimates in terms of non-homogeneous terms.

For any $B_r(x_0) \subset B_1$, write the equation as

$$\int_{B_1} a^{ij}(x_0) D_i u D_j \varphi\,dx = \int_{B_1} f\varphi - cu\varphi + (a^{ij}(x_0) - a^{ij}(x)) D_i u D_j \varphi\,dx.$$

In $B_r(x_0)$, the Dirichlet problem,

$$\int_{B_r(x_0)} a^{ij}(x_0) D_i w D_j \varphi\,dx = 0 \quad \text{for any} \quad \varphi \in H_0^1(B_r(x_0))$$

has a unique weak solution in $H_0^1(B_r(x_0))$ and $u - w \in H_0^1(B_r(x_0))$. Clearly, $v = u - w$ belongs in $H_0^1(B_r(x_0))$ and satisfies

$$\int_{B_1} a^{ij}(x_0) D_i v D_j \varphi\,dx = \int_{B_1} f\varphi - cu\varphi + (a^{ij}(x_0) - a^{ij}(x)) D_i u D_j \varphi\,dx \tag{3.85}$$

for any $\varphi \in H_0^1(B_r(x_0))$. By taking the test function $\varphi = v$, we have the following estimates on each term in the right-hand side of (3.85):

$$\begin{aligned}\int_{B_r(x_0)} fv\,dx &\le \left(\int_{B_r(x_0)} f^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{2n}} \left(\int_{B_r(x_0)} |v|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{2n}} \\ &\le \left(\int_{B_r(x_0)} f^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{2n}} \left(\int_{B_r(x_0)} |Dv|^2\,dx\right)^{\frac{1}{2}},\end{aligned}$$

$$\begin{aligned}\int_{B_r(x_0)} cuv\,dx &\le \left(\int_{B_r(x_0)} |c|^n\,dx\right)^{\frac{1}{n}} \left(\int_{B_r(x_0)} |uv|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \\ &\le \left(\int_{B_r(x_0)} |c|^n\,dx\right)^{\frac{1}{n}} \left(\int_{B_r(x_0)} |u|^2\,dx\right)^{\frac{1}{2}} \left(\int_{B_r(x_0)} |v|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{2n}} \\ &\le \left(\int_{B_r(x_0)} |c|^n\,dx\right)^{\frac{1}{n}} \left(\int_{B_r(x_0)} |u|^2\,dx\right)^{\frac{1}{2}} \left(\int_{B_r(x_0)} |Dv|^2\,dx\right)^{\frac{1}{2}},\end{aligned}$$

$$\int_{B_r(x_0)} (a^{ij}(x_0) - a^{ij}(x))D_iuD_jv\,dx \le \tau(r)\left(\int_{B_r(x_0)} |Du|^2\,dx\right)^{\frac{1}{2}} \left(\int_{B_r(x_0)} |Dv|^2\,dx\right)^{\frac{1}{2}},$$

where we used Hölder's inequality and the Sobolev embedding theorem. From the uniform ellipticity condition, we estimate the terms in (3.85) by using the previous three estimates then divide both sides of the inequality by $\|Dv\|_{L^2(B_r(x_0))}$ to get

$$\begin{aligned}&\int_{B_r(x_0)} |Dv|^2\,dx \\ &\quad\le C\left\{\tau(r)^2\int_{B_r(x_0)} |Du|^2\,dx + \left(\int_{B_r(x_0)} |c|^n\,dx\right)^{2/n}\int_{B_r(x_0)} |u|^2\,dx + \left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{n}}\right\}.\end{aligned}$$

Therefore, Lemma 3.12 implies that for any $0 < \rho \le r$,

$$\begin{aligned}\int_{B_\rho(x_0)} |Du|^2\,dx \le C\Bigg\{&\left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right)\int_{B_r(x_0)} |Du|^2\,dx \\ &+ \left(\int_{B_r(x_0)} |c|^n\,dx\right)^{2/n}\int_{B_r(x_0)} |u|^2\,dx + \left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{n}}\Bigg\}, \quad (3.86)\end{aligned}$$

where $C = (n, \lambda, \Lambda)$ is a positive constant. By Hölder's inequality,

$$\left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{n}} \le \left(\int_{B_r(x_0)} |f|^q\,dx\right)^{\frac{2}{q}} r^{n-2+2\alpha},$$

where $\alpha = 2 - \frac{n}{q} \in (0,1)$ whenever $q \in (\frac{n}{2}, n)$. Thus, (3.86) implies for any $B_r(x_0) \subset B_1$ and any $0 < \rho \le r$,

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\Bigg\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right) \int_{B_r(x_0)} |Du|^2\,dx + \left(\int_{B_r(x_0)} |c|^n\,dx\right)^{2/n} \int_{B_r(x_0)} |u|^2\,dx + r^{n-2+2\alpha}\|f\|^2_{L^q(B_1)}\Bigg\}.$$

Case 1: $c \equiv 0$.

We have for any $B_r(x_0) \subset B_1$ and for any $0 < \rho \le r$,

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\Bigg\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right) \int_{B_r(x_0)} |Du|^2\,dx + r^{n-2+2\alpha}\|f\|^2_{L^q(B_1)}\Bigg\}.$$

By Lemma 3.9, we may replace $r^{n-2+2\alpha}$ in the last estimate by $\rho^{n-2+2\alpha}$, in which case the proof is complete. More precisely, there exists an $R_0 > 0$ such that for any $x_0 \in B_{1/2}(0)$ and any $0 < \rho < r \le R_0$, we have

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\Bigg\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right) \int_{B_r(x_0)} |Du|^2\,dx + \rho^{n-2+2\alpha}\|f\|^2_{L^q(B_1)}\Bigg\}.$$

In particular, taking $r = R_0$ yields for any $\rho < R_0$,

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\rho^{n-2+2\alpha}\Bigg\{ \int_{B_1} |Du|^2\,dx + \|f\|^2_{L^q(B_1)}\Bigg\}.$$

Case 2: General coefficient $c \in L^n(B_1)$. We have for any $B_r(x_0) \subset B_1$ and any $0 < \rho \le r$,

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\Bigg\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right) \int_{B_r(x_0)} |Du|^2\,dx + r^{n-2+2\alpha}\chi(F) + \int_{B_r(x_0)} u^2\,dx\Bigg\} \tag{3.87}$$

where $\chi(F) = \|f\|^2_{L^q(B_1)}$. We will prove, via a bootstrap argument, that for any $x_0 \in B_{1/2}$ and any $0 < \rho < r < 1/2$,

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\Bigg\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right) \int_{B_r(x_0)} |Du|^2\,dx + r^{n-2+2\alpha}\left(\chi(F) + \int_{B_1} u^2\,dx + \int_{B_1} |Du|^2\,dx\right)\Bigg\}. \tag{3.88}$$

First by Lemma 3.10, there exists an $R_1 \in (1/2, 1)$ such that there holds for any $x_0 \in B_{R_1}$ and any $0 < r \le 1 - R_1$

$$\int_{B_r(x_0)} u^2\,dx \le Cr^{\delta_1}\Bigg\{ \int_{B_1} |Du|^2\,dx + \int_{B_1} u^2\,dx\Bigg\} \tag{3.89}$$

where $\delta_1 = 2$ if $n > 2$ and δ_1 is arbitrary in $(0, 2)$ if $n = 2$. This, combined with (3.87), implies

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\left\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2 \right) \int_{B_r(x_0)} |Du|^2\,dx + r^{n-2+2\alpha}\chi(F) + r^{\delta_1}\|u\|^2_{H^1(B_1)} \right\}.$$

Then (3.88) holds in the following cases:

(i) $n = 2$, by choosing $\delta_1 = 2\alpha$,

(ii) $n > 2$ while $n - 2 + 2\alpha \le 2$, by choosing $\delta_1 = 2$.

However, for $n > 2$ and $n - 2 + 2\alpha > 2$, we have

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\left\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2 \right) \int_{B_r(x_0)} |Du|^2\,dx + r^2 \left(\chi(F) + r^{\delta_1}\|u\|^2_{H^1(B_1)} \right) \right\}.$$

□

Lemma 3.9 again implies that for any $x_0 \in B_{R_1}$ and any $0 < r \le 1 - R_1$

$$\int_{B_r(x_0)} |Du|^2\,dx \le Cr^2 \left\{ \chi(F) + \|u\|^2_{H^1(B_1)} \right\}.$$

Then by Lemma 3.10, there exists an $R_2 \in (1/2, R_1)$ such that there holds for any $x_0 \in B_{R_2}$ and any $0 < r \le R_1 - R_2$

$$\int_{B_r(x_0)} u^2\,dx \le Cr^{\delta_2} \left\{ \chi(F) + \|u\|^2_{H^1(B_1)} \right\} \tag{3.90}$$

where $\delta_2 = 4$ if $n > 4$ and δ_2 is arbitrary in $(2, n)$ if $n = 3$ or 4. Notice that this last estimate (3.90) is an improvement compared with (3.89). Substitute (3.90) in (3.87) and continue the process. After finite steps we arrive at (3.88).

3.6.4 Hölder Continuity of the Gradient

As before, we take $U = B_1$. We have the following estimate for the gradient of weak solutions of equation (3.81). The proof is similar as before, so we omit the details.

Theorem 3.26. *Let $u \in H^1(B_1)$ be a weak solution of (3.81). Assume $a^{ij} \in C^\alpha(\bar{B}_1)$, $c \in L^q(B_1)$ and $f \in L^q(B_1)$ for some $q > n$ and $\alpha = 1 - n/q \in (0, 1)$. Then $Du \in C^\alpha(B_1)$. Moreover, there exists an $R_0 = R_0(\lambda, |a^{ij}|_{C^\alpha}, \|c\|_{L^q})$ such that for any $x_0 \in B_{1/2}$ and $r \le R_0$, there holds*

$$\int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2\,dx \le Cr^{n+2\alpha} \left\{ \|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)} \right\},$$

where $C = C(\lambda, |a^{ij}|_{C^\alpha}, \|c\|_{L^q})$ is a positive constant.

3.7 De Giorgi–Nash–Moser Regularity Theory

This section introduces the celebrated De Giorgi–Nash–Moser regularity theory for the Hölder continuity of solutions, and we introduce two ideas for completeness. That is, we first introduce De Giorgi's approach which develops the local boundedness of solutions followed by the estimate on its oscillation. These two ingredients will imply the Hölder continuity of solutions. Then, we study Moser's approach, which also establishes the same local boundedness result combined with his version of a Harnack inequality to conclude the same result on the Hölder continuity of solutions. Note carefully that, unlike in the previous section, we will not make any regularity assumptions on the coefficients of the elliptic operators. Furthermore, the overall idea we use here relies on a delicate iteration technique rather than perturbation methods.

3.7.1 Motivation

Before we proceed with the technical aspects of this theory, let us motivate its historical relevance. The renowned nineteenth problem in Hilbert's famous program asked whether or not minimizers of the energy functional

$$J(w) = \int_U L(Dw)\,dx \quad \text{for} \quad w \in H_0^1(U) \cap H^2(U),$$

are smooth. Here, the Lagrangian L is assumed to be smooth and satisfies some additional conditions (such as those described in Theorem 2.23 of Chapter 2). The Euler-Lagrange equation for this variational problem is the elliptic equation

$$\sum_{i=1}^{n} (L_{p_i}(Dw))_{x_i} = 0 \quad \text{in } U. \tag{3.91}$$

In fact, the minimizers can be easily shown to be smooth using the Schauder estimates and a standard bootstrap argument but at the expense of requiring the minimizer be of class $C^{1,\alpha}$ a priori. The main result of the De Giorgi–Nash–Moser theory precisely ensures this initial regularity holds true and thus providing the crucial ingredient in resolving Hilbert's nineteenth problem.

What follows is only a rough explanation of the procedure but the arguments can certainly be made rigorous. If we formally differentiate equation (3.91) with respect to x_k then insert (3.91) into the resulting calculation, we would obtain

$$\sum_{i,j=1}^{n} (L_{p_i p_j}(Dw) w_{x_j x_k})_{x_i} = 0.$$

Thus, if we set $u = w_{x_k}$, this implies that u satisfies the linear elliptic equation

$$\sum_{i,j=1}^{n} (a^{ij}(x) u_{x_j})_{x_i} = 0, \tag{3.92}$$

where $a^{ij}(x) = L_{p_ip_j}(Dw(x))$ satisfies some type of uniform ellipticity condition. De Giorgi–Nash–Moser theory states that if u is a weak solution of equation (3.92), then u is Hölder continuous and so w is a $C^{1,\alpha}$ solution of (3.91). Hence, the coefficients $a^{ij}(x)$ are Hölder continuous and the Schauder estimates imply that $u \in C^{2,\alpha}$. By bootstrap, u is of class $C^{k,\alpha}$ for $k = 2, 3, 4, \ldots$ and is therefore, along with w, smooth.

3.7.2 Local Boundedness and Preliminary Lemmas

Both De Giorgi and Moser's approach rely initially on the local boundedness of solutions before arriving at the Hölder regularity result. We now state this result but defer its proof until the next section.

Theorem 3.27 (local boundedness)**.** *Suppose $a^{ij} \in L^\infty(B_1)$ and $c \in L^q(B_1)$ for some $q > n/2$ satisfy the following assumptions:*

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \;\; \textit{for any} \;\; x \in B_1, \; \xi \in \mathbb{R}^n,$$

and

$$\|a^{ij}\|_{L^\infty(B_1)} + \|c\|_{L^q(B_1)} \leq \Lambda$$

for some positive constants λ and Λ. Suppose that $u \in H^1(B_1)$ is a sub-solution in the following sense:

$$\int_{B_1} a^{ij}D_iuD_j\varphi + cu\varphi\,dx \leq \int_{B_1} f\varphi\,dx \;\; \textit{for any non-negative} \; \varphi \in H^1_0(B_1). \tag{3.93}$$

If $f \in L^q(B_1)$, then $u^+ \in L^\infty_{loc}(B_1)$. Moreover, there holds for any $\theta \in (0,1)$ and $p > 0$

$$\sup_{B_\theta} u^+ \leq C\Big\{\frac{1}{(1-\theta)^{n/p}}\|u^+\|_{L^p(B_1)} + \|f\|_{L^q(B_1)}\Big\},$$

where $C = C(n, \lambda, \Lambda, p, q)$ is a positive constant.

One strategy to prove this is to use a clever iteration procedure of Moser, which will also appear in our proof of the weak Harnack inequality below. In either case, Moser's iteration procedure will also make use of the following elementary result.

Lemma 3.13. *Let U be a bounded subset, $u : U \mapsto \mathbb{R}$ is measurable, $|u|^p \in L^1(U)$ for $p \geq 1$ and assume*

$$\Phi(p) := \Big(\frac{1}{|U|}\int_U |u|^p\,dx\Big)^{1/p}$$

is well-defined. Then

$$\lim_{p\to\infty} \Phi(p) = \sup_U |u|.$$

Proof. Let $p' > p$ be arbitrary and we may assume u to be non-negative. If $u \in L^{p'}(U)$, then Hölder's inequality yields

$$\begin{aligned}\Big(\frac{1}{|U|}\int_U u^p\,dx\Big)^{1/p} &\le \frac{1}{|U|^{1/p}}\Big(\int_U 1\,dx\Big)^{\frac{p'-p}{pp'}}\Big(\int_U (u^p)^{p'/p}\,dx\Big)^{1/p'}\\ &= \Big(\frac{1}{|U|}\int_U u^{p'}\,dx\Big)^{1/p'}.\end{aligned}$$

Hence, $\Phi(p)$ is monotone increasing with respect to $p > 1$. Moreover, $\Phi(p)$ is bounded above by $\sup_U u$ since

$$\Phi(p) \le \Big(\frac{1}{|U|}\int_U (\sup_U u)^p\,dx\Big)^{1/p} \le \sup_U u.$$

Thus, $\lim_{p\to\infty}\Phi(p)$ exists and $\lim_{p\to\infty}\Phi(p) \le \sup_U u$.

On the other hand, by definition of the essential supremum, for each $\epsilon > 0$ there exists $\delta > 0$ such that $|A| > \delta|U|$, where

$$A = \{x \in U \mid u(x) \ge \sup_U u - \epsilon\}.$$

Therefore,

$$\Phi(p) \ge \Big(\frac{|A|}{|U|}\Big)^{1/p}\Big(\frac{1}{|A|}\int_A u^p\,dx\Big)^{1/p} \ge \delta^{1/p}(\sup_U u - \epsilon).$$

Hence, after sending $p \longrightarrow \infty$ we get

$$\liminf_{p\to\infty}\Phi(p) \ge \sup u - \epsilon \quad \text{for every } \epsilon > 0.$$

Both set of estimates imply that $\lim_{p\to\infty}\Phi(p) = \sup u$. This completes the proof of the lemma. □

After establishing local boundedness, the Hölder continuity of weak solutions will be a consequence of the following important lemma and a Harnack or oscillation inequality.

Lemma 3.14. *Let ω and σ be non-decreasing functions in an interval $(0, R]$. Suppose there holds for all $r \le R$,*

$$\omega(\tau r) \le \gamma\omega(r) + \sigma(r)$$

for some $0 < \gamma, \tau < 1$. Then for any $\mu \in (0, 1)$ and $r \le R$ we have

$$\omega(r) \le C\Big\{\Big(\frac{r}{R}\Big)^\alpha \omega(R) + \sigma(r^\mu R^{1-\mu})\Big\}$$

where $C = C(\gamma, \tau)$ is a positive constant and $\alpha = (1-\mu)\log\gamma/\log\tau$.

Proof. Fix some $r_1 \le R$. Then for any $r \le r_1$ we have

$$\omega(\tau r) \le \gamma\omega(r) + \sigma(r_1)$$

since σ is non-decreasing. We now iterate this inequality to get for any positive integer k

$$\omega(\tau^k r_1) \le \gamma^k \omega(r_1) + \sigma(r_1)\sum_{i=0}^{k-1}\gamma^i \le \gamma^k\omega(R) + \frac{\sigma(r_1)}{1-\gamma}.$$

For any $r \le r_1$, choose k so that

$$\tau^k r_1 < r \le \tau^{k-1} r_1.$$

This ensures that $(\log\gamma^k)(\log\tau) \le (\log\gamma)(\log(r/r_1))$ and so

$$\gamma^k \le (r/r_1)^{\log\gamma/\log\tau}.$$

Hence, the monotonicity of ω then implies that

$$\omega(r) \le \omega(\tau^{k-1}r_1) \le \gamma^{k-1}\omega(R) + \frac{\sigma(r_1)}{1-\gamma} \le \frac{1}{\gamma}\Big(\frac{r}{r_1}\Big)^{\frac{\log\gamma}{\log\tau}}\omega(R) + \frac{\sigma(r_1)}{1-\gamma}.$$

If we take $r_1 = r^\mu R^{1-\mu}$, we obtain

$$\omega(r) \le \frac{1}{\gamma}\Big(\frac{r}{R}\Big)^{(1-\mu)\frac{\log\gamma}{\log\tau}}\omega(R) + \frac{\sigma(r^\mu R^{1-\mu})}{1-\gamma}.$$

□

3.7.3 Proof of Local Boundedness: Moser Iteration

To illustrate the main idea in our proof of Theorem 3.27, let us describe our strategy for the case when $f \equiv 0$, $\theta = 1/2$ and $p = 2$. By choosing an appropriate test function, we will estimate the L^{p_1} norm of u in a smaller ball by the L^{p_2} norm of u in a larger ball for $p_1 > p_2$; that is, we establish a reverse type Hölder inequality

$$\|u\|_{L^{p_1}(B_{r_1})} \le C\|u\|_{L^{p_2}(B_{r_2})}, \tag{3.94}$$

for $p_1 > p_2$ and $r_1 < r_2$. The issue is our choice of test function forces the constant C to behave like $(r_2 - r_1)^{-1}$. Moser's approach, however, is to carefully iterate the estimate and choose sequences $\{r_i\}$ and $\{p_i\}$ which avoids this constant from blowing up. Thus, this iteration technique and Lemma 3.13 allows us to send $p_1 \to \infty$, $p_2 \to 2$, $r_1 \to 1/2$ and $r_2 \to 1$ in (3.94) to get the desired estimate.

Proof of Theorem 3.27. The proof is long, so we divide it into several steps.

Step 1: We prove the theorem for $\theta = 1/2$ and $p = 2$. We follow Moser's proof, but an alternative proof by De Giorgi can also be found in [16]. For some $k > 0$ and $m > 0$, set $\bar{u} = u^+ + k$ and

$$\bar{u}_m = \begin{cases} \bar{u} & \text{if } u < m, \\ m + k & \text{if } u \geq m. \end{cases}$$

Then we have $D\bar{u}_m = 0$ in $\{u < 0\}$ and $\{u > m\}$ and $\bar{u}_m \leq \bar{u}$. Set the test function

$$\varphi = \eta^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \in H_0^1(B_1)$$

for some $\beta \geq 0$ and some non-negative function $\eta \in C_0^1(B_1)$. Direct calculation yields

$$\begin{aligned} D\varphi &= \beta\eta^2\bar{u}_m^{\beta-1} D\bar{u}_m \bar{u} + \eta^2 \bar{u}_m^\beta D\bar{u} + 2\eta D\eta(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \\ &\geq \eta^2 \bar{u}_m^\beta(\beta D\bar{u}_m + D\bar{u}) + 2\eta D\eta(\bar{u}_m^\beta \bar{u} - k^{\beta+1}). \end{aligned} \tag{3.95}$$

Note that $\varphi = 0$ and $D\varphi = 0$ in $\{u \leq 0\}$. Hence, if we substitute such φ in the equation, we integrate in the set $\{u > 0\}$ then send m to infinity. Note also that $u^+ \leq \bar{u}$ and $\bar{u}_m^\beta \bar{u} - k^{\beta+1} \leq \bar{u}_m^\beta \bar{u}$ for $k > 0$. From the elementary inequality $ab \leq 2ab \leq a^2 + b^2$ for $a, b \geq 0$, we have

$$\begin{aligned} \Lambda|D\bar{u}||D\eta|\bar{u}_m^\beta \bar{u}\eta &= a \times b \\ &:= \Lambda(2/\lambda)^{1/2}|D\eta|\bar{u}_m^{\beta/2}\bar{u} \times (\lambda/2)^{1/2}\eta\bar{u}_m^{\beta/2}|D\bar{u}| \\ &\leq \frac{2\Lambda^2}{\lambda}|D\eta|^2\bar{u}_m^\beta\bar{u}^2 + \frac{\lambda}{2}\eta^2\bar{u}_m^\beta|D\bar{u}|^2. \end{aligned} \tag{3.96}$$

Hence,

$$\begin{aligned} \int a^{ij}(x)D_i u D_j\varphi\,dx &= \int a^{ij}(x)D_i\bar{u}(\beta D_j\bar{u}_m + D_j\bar{u})\eta^2\bar{u}_m^\beta + 2\int a^{ij}(x)D_i\bar{u}D_j\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1})\eta\,dx \\ &\geq \lambda\beta\int \eta^2\bar{u}_m^\beta|D\bar{u}_m|^2\,dx + \lambda\int \eta^2\bar{u}_m^\beta|D\bar{u}|^2\,dx - \Lambda\int |D\bar{u}||D\eta|\bar{u}_m^\beta\bar{u}\eta\,dx \\ &\geq \lambda\beta\int \eta^2\bar{u}_m^\beta|D\bar{u}_m|^2\,dx + \frac{\lambda}{2}\int \eta^2\bar{u}_m^\beta|D\bar{u}|^2\,dx - \frac{2\Lambda^2}{\lambda}\int |D\eta|^2\bar{u}_m^\beta\bar{u}^2\,dx, \end{aligned}$$

where we used (3.95) in the first line and we used (3.96) to estimate the last line. Therefore, noting that $\bar{u} \geq k$, we obtain

$$\begin{aligned} \beta\int \eta^2\bar{u}_m^\beta|D\bar{u}_m|^2\,dx + \int \eta^2\bar{u}_m^\beta|D\bar{u}|^2\,dx &\leq C\left\{\int |D\eta|^2\bar{u}_m^\beta\bar{u}^2\,dx + \int |c|\eta^2\bar{u}_m^\beta\bar{u}^2 + |f|\eta^2\bar{u}_m^\beta\bar{u}\,dx\right\} \\ &\leq C\left\{\int |D\eta|^2\bar{u}_m^\beta\bar{u}^2\,dx + \int c_0\eta^2\bar{u}_m^\beta\bar{u}^2\,dx\right\}, \end{aligned} \tag{3.97}$$

where c_0 is defined as

$$c_0 = |c| + \frac{|f|}{k}.$$

Choose $k = \|f\|_{L^q(B_1)}$ if f is not identically 0. Otherwise, choose arbitrary $k > 0$ and eventually let $k \longrightarrow 0^+$. By assumption, we have

$$\|c_0\|_{L^q} \leq \Lambda + 1.$$

Set $w = \bar{u}_m^{\beta/2}\bar{u}$ and so

$$|Dw|^2 \leq (1+\beta)\Big\{\beta\bar{u}_m^\beta|D\bar{u}_m|^2 + \bar{u}_m^\beta|D\bar{u}|^2\Big\}.$$

Thus, from (3.97) we have

$$\int |Dw|^2\eta^2\,dx \leq C\left\{(1+\beta)\int w^2|D\eta|^2\,dx + (1+\beta)\int c_0w^2\eta^2\,dx\right\}$$

or

$$\int |D(w\eta)|^2\eta^2\,dx \leq C\left\{(1+\beta)\int w^2|D\eta|^2\,dx + (1+\beta)\int c_0w^2\eta^2\,dx\right\}. \tag{3.98}$$

Hölder's inequality implies

$$\int c_0w^2\eta^2\,dx \leq \left(\int c_0^q\,dx\right)^{\frac{1}{q}}\left(\int (\eta w)^{\frac{2q}{q-1}}\,dx\right)^{1-1/q} \leq (1+\Lambda)\left(\int (\eta w)^{\frac{2q}{q-1}}\,dx\right)^{1-1/q}.$$

By interpolation and Sobolev embedding with $2^* = \frac{2n}{n-2} > \frac{2q}{q-1} > 2$ if $q > \frac{n}{2}$, we have

$$\begin{aligned}\|\eta w\|_{L^{\frac{2q}{q-1}}} &\leq \epsilon\|\eta w\|_{L^{2^*}} + C(n,q)\epsilon^{-\frac{n}{2q-n}}\|\eta w\|_{L^2}\\ &\leq \epsilon\|D(\eta w)\|_{L^2} + C(n,q)\epsilon^{-\frac{n}{2q-n}}\|\eta w\|_{L^2}\end{aligned}$$

for small $\epsilon > 0$. Therefore, combining this with (3.98) yields

$$\int |D(w\eta)|^2\,dx \leq C\left\{(1+\beta)\int w^2|D\eta|^2\,dx + (1+\beta)^{\frac{2q}{2q-n}}\int w^2\eta^2\,dx\right\},$$

and in particular

$$\int |D(w\eta)|^2\,dx \leq C(1+\beta)^\alpha\int (|D\eta|^2+\eta^2)w^2\,dx,$$

where α is a positive number depending only on n and q. Sobolev embedding then implies

$$\left(\int |\eta w|^{2\chi}\,dx\right)^{1/\chi} \leq C(1+\beta)^\alpha\int (|D\eta|^2+\eta^2)w^2\,dx,$$

where $\chi = \frac{n}{n-2} > 1$ for $n > 2$ and $\chi > 2$ for $n = 2$.

Choose the cutoff function as follows. For any $0 < r < R \le 1$, set $\eta \in C_0^1(B_R)$ with the property

$$\eta \equiv 1 \text{ in } B_r \text{ and } |D\eta| \le \frac{2}{R-r}.$$

Then we obtain

$$\left(\int_{B_r} w^{2\chi}\, dx\right)^{1/\chi} \le C\frac{(1+\beta)^\alpha}{(R-r)^2}\int_{B_R} w^2\, dx.$$

If we recall the definition of w, we have

$$\left(\int_{B_r} \bar{u}^{2\chi}\bar{u}_m^{\beta\chi}\, dx\right)^{1/\chi} \le C\frac{(1+\beta)^\alpha}{(R-r)^2}\int_{B_R} \bar{u}^2\bar{u}_m^{\beta}\, dx.$$

Set $\gamma = \beta + 2 \ge 2$, then we get

$$\left(\int_{B_r} \bar{u}_m^{\gamma\chi}\, dx\right)^{1/\chi} \le C\frac{(\gamma-1)^\alpha}{(R-r)^2}\int_{B_R} \bar{u}^{\gamma}\, dx$$

provided that the integral on the right-hand side is finite. By sending $m \longrightarrow \infty$, we conclude that

$$\|\bar{u}\|_{L^{\gamma\chi}(B_r)} \le \left(C\frac{(\gamma-1)^\alpha}{(R-r)^2}\right)^{1/\gamma}\|\bar{u}\|_{L^{\gamma}(B_R)}$$

provided that $\|\bar{u}\|_{L^\gamma(B_R)} < \infty$, where $C = C(n, q, \lambda, \Lambda)$ is a positive constant independent of γ. We shall iterate the previous estimated beginning with $\gamma = 2$ and proceed via $2, 2\chi, 2\chi^2, \ldots$. Now set for $i = 0, 1, 2, \ldots$,

$$\gamma_i = 2\chi^i \text{ and } r_i = \frac{1}{2} + \frac{1}{2^{i+1}}.$$

Since $\gamma_i = \chi\gamma_{i-1}$ and $r_{i-1} - r_i = 2^{-(i+1)}$, we have for $i = 1, 2, \ldots$,

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \le C(n, q, \lambda, \Lambda)^{\frac{i}{\chi^{i-1}}}\|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})}$$

provided that $\|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_i-1})} < \infty$. Hence, by iteration, we obtain

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \le C^{\sum\frac{i}{\chi^{i-1}}}\|\bar{u}\|_{L^2(B_1)}$$

or in particular,

$$\left(\int_{B_{r_i}} \bar{u}^{2\chi^i}\, dx\right)^{\frac{1}{2\chi^i}} \le C^{\sum\frac{i}{\chi^{i-1}}}\left(\int_{B_1}\bar{u}^2\, dx\right)^{\frac{1}{2}}.$$

Sending $i \longrightarrow \infty$ in the previous estimate yields

$$\sup_{B_{1/2}} \bar{u} \le C\|\bar{u}\|_{L^2(B_1)} \text{ or } \sup_{B_{1/2}} u^+ \le C\|u^+\|_{L^2(B_1)} + k = C\Big\{\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\Big\}.$$

This completes the proof of the theorem for the case $p = 2$.

Remark 3.14. *If the subsolution u is bounded, we may simply take the test function*

$$\varphi = \eta^2(\bar{u}^{\beta+1} - k^{\beta+1}) \in H_0^1(B_1).$$

for some $\beta \geq 0$ and some non-negative function $\eta \in C_0^1(B_1)$.

Step 2: We now prove the theorem for $p \geq 2$.

Based on a dilation argument, we take any $R \leq 1$ and define

$$\tilde{u}(y) = u(Ry) \text{ for } y \in B_1.$$

It is easy to see that $\tilde{u}$ satisfies

$$\int_{B_1} \tilde{a}^{ij}(x) D_i\tilde{u}\, D_j\varphi + \tilde{c}\tilde{u}\varphi\, dx \leq \int_{B_1} \tilde{f}\varphi\, dx$$

for any non-negative $\varphi \in H_0^1(B_1)$ where

$$\tilde{a}(y) = a(Ry),\ \tilde{c}(y) = R^2 c(Ry), \text{ and } \tilde{f}(y) = R^2 f(Ry),$$

for any $y \in B_1$. Direct calculation shows

$$\|\tilde{a}^{ij}\|_{L^\infty(B_1)} + \|\tilde{c}\|_{L^q(B_1)} = \|\tilde{a}^{ij}\|_{L^\infty(B_1)} + R^{2-n/q}\|c\|_{L^q(B_R)} \leq \Lambda.$$

We may apply what we proved above to $\tilde{u}$ in B_1 (iterating with $\gamma = p$ instead of $\gamma = 2$) and rewrite the result in terms of u. Hence, we obtain for $p \geq 2$

$$\sup_{B_{R/2}} u^+ \leq C\left\{ R^{-n/p}\|u^+\|_{L^p(B_R)} + R^{2-n/q}\|f\|_{L^q(B_R)} \right\}$$

where $C = C(n, \lambda, \Lambda, p, q)$ is a positive constant. The estimate in $B_{\theta R}$ can be obtained by applying the above result to $B_{(1-\theta)R}(y)$ for any $y \in B_{\theta R}$. Take $R = 1$. This is Theorem 3.27 for any $\theta \in (0, 1)$ and $p \geq 2$.

Step 3: We now prove the theorem for $p \in (0, 2)$. We show that for any $\theta \in (0, 1)$ and $0 < R \leq 1$ there holds

$$\begin{aligned}\|u^+\|_{L^\infty(B_{\theta R})} &\leq C\left\{ \frac{1}{[(1-\theta)R]^{n/2}}\|u^+\|_{L^2(B_R)} + R^{2-n/q}\|f\|_{L^q(B_R)} \right\} \\ &\leq C\left\{ \frac{1}{[(1-\theta)R]^{n/2}}\|u^+\|_{L^2(B_R)} + \|f\|_{L^q(B_R)} \right\}.\end{aligned}$$

For $p \in (0, 2)$ we have

$$\int_{B_R} (u^+)^2\, dx \leq \|u^+\|_{L^\infty(B_R)}^{2-p} \int_{B_R} (u^+)^p\, dx.$$

Thus, by Hölder's inequality,

$$\begin{aligned}\|u^+\|_{L^\infty(B_{\theta R})} &\le C\left\{\frac{1}{[(1-\theta)R]^{n/2}}\|u^+\|^{1-p/2}_{L^\infty(B_R)}\left(\int_{B_R}(u^+)^p\,dx\right)^{\frac{1}{2}}+\|f\|_{L^q(B_R)}\right\}\\ &\le \frac{1}{2}\|u^+\|_{L^\infty(B_R)}+C\left\{\frac{1}{[(1-\theta)R]^{n/p}}\left(\int_{B_R}(u^+)^p\,dx\right)^{\frac{1}{p}}+\|f\|_{L^q(B_R)}\right\}.\end{aligned}$$

Set $h(t)=\|u^+\|_{L^\infty(B_t)}$ for $t\in(0,1]$ so that the previous estimate can be rewritten as

$$h(r)\le\frac{1}{2}h(R)+\frac{C}{(R-r)^{n/p}}\|u^+\|_{L^p(B_1)}+C\|f\|_{L^q(B_1)}\ \text{ for any } 0<r<R\le 1.$$

We apply Lemma 3.15 from below to get for any $0<r<R<1$

$$h(r)\le\frac{C}{(R-r)^{n/p}}\|u^+\|_{L^p(B_1)}+C\|f\|_{L^q(B_1)}.$$

Let $R\longrightarrow 1^-$. Hence, for any $0<\theta<1$ we get the desired estimate

$$\|u^+\|_{L^\infty(B_\theta)}\le\frac{C}{(1-\theta)^{n/p}}\|u^+\|_{L^p(B_1)}+C\|f\|_{L^q(B_1)}.$$

□

At the end of the proof, recall that we invoked the following lemma whose proof can be found in [16].

Lemma 3.15. *Let $h(t)\ge 0$ be bounded in $[\tau_0,\tau_1]$ with $\tau_0\ge 0$. Suppose for $\tau_0\le t<s\le\tau_1$ we have*

$$h(t)\le\theta h(s)+\frac{A}{(s-t)^\alpha}+B$$

for some $\theta\in[0,1)$. Then for any $\tau_0\le t<s\le\tau_1$ there holds

$$h(t)\le c(\alpha,\theta)\left\{\frac{A}{(s-t)^\alpha}+B\right\}.$$

Moser's iteration can again be applied to prove a closely related high integrability result. We omit its proof but refer the reader to [16] for the details.

Theorem 3.28 (high integrability)**.** *Suppose $a^{ij}\in L^\infty(B_1)$ and $c\in L^{\frac{n}{2}}(B_1)$ satisfy the following assumption:*

$$\lambda|\xi|^2\le a^{ij}(x)\xi_i\xi_j\le\Lambda|\xi|^2\ \textit{for any}\ x\in B_1,\xi\in\mathbb{R}^n,$$

for some positive constants λ and Λ. Suppose that $u \in H^1(B_1)$ is a subsolution in the following sense:

$$\int_{B_1} a^{ij}(x) D_i u \, D_j \varphi + c(x) u \varphi \, dx \leq \int_{B_1} f \varphi \, dx$$

for any non-negative $\varphi \in H_0^1(B_1)$. If $f \in L^q(B_1)$ for some $q \in [\frac{2n}{n+2}, \frac{n}{2})$, then $u^+ \in L^{q^*}_{loc}(B_1)$ for $\frac{1}{q^*} = \frac{1}{q} - \frac{2}{n}$. Moreover, there holds

$$\|u^+\|_{L^{q^*}(B_{1/2})} \leq C \Big\{ \|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)} \Big\}$$

where $C = C(n, \lambda, \Lambda, q, \epsilon(K))$ is a positive constant with

$$\epsilon(K) = \left(\int_{\{|c|>K\}} |c|^{\frac{n}{2}} \right)^{\frac{2}{n}}.$$

3.7.4 Hölder Regularity: De Giorgi's Approach

For simplicity, we establish the Hölder continuity of weak solutions to homogeneous equations without lower-order terms,

$$Lu \equiv - \sum_{i,j=1}^{n} D_i \left(a^{ij}(x) D_j u \right) \text{ in } B_1(0) \subset \mathbb{R}^n,$$

where $a^{ij} \in L^\infty(B_1)$ satisfies

$$\lambda |\xi|^2 \leq a^{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \text{ for all } x \in B_1(0) \text{ and } \xi \in \mathbb{R}^n$$

for some positive constants λ and Λ.

Definition 3.10. *The function $u \in H^1_{loc}(B_1)$ is called a* **subsolution** *(resp.* **supersolution***) of the equation $Lu = 0$ if*

$$\int_{B_1} a^{ij}(x) D_i u \, D_j \varphi \, dx \leq 0 \ (\textit{resp.} \geq 0) \ \textit{for all non-negative} \ \varphi \in H_0^1(B_1).$$

First, we will need the following, which indicates that monotone convex mappings preserve subsolutions and supersolutions. The proof follows from a direct computation and we omit the details (cf. [16] for the proof).

Lemma 3.16. *Let $\Phi \in C^{0,1}_{loc}(\mathbb{R})$ be convex. Then*

(i) If u is a subsolution and $\Phi' \geq 0$, then $v = \Phi(u)$ is also a subsolution provided that $v \in H^1_{loc}(B_1)$.

(ii) If u is a supersolution and $\Phi' \leq 0$, then $v = \Phi(u)$ is a subsolution provided that $v \in H^1_{loc}(B_1)$.

Next is a Poincaré type inequality. But unlike the more common Poincaré inequalities that assume u belongs to $H_0^1(B_1)$ or an inequality that involves the difference between u and its average, this version says that if $u \in H^1(B_1)$ vanishes in a measurable portion of the domain, then it can be controlled by its gradient in L^2.

Lemma 3.17 (Poincaré–Sobolev)**.** *For any $\epsilon > 0$ there exists a constant $C = C(\epsilon, n)$ such that for $u \in H^1(B_1)$ with $\mu(\{x \in B_1 \,|\, u = 0\}) \geq \epsilon\mu(B_1)$, there holds*

$$\int_{B_1} u^2\,dx \leq C\int_{B_1} |Du|^2\,dx.$$

Proof. Suppose the contrary. Then there is a sequence $\{u_m\} \subset H^1(B_1)$ such that

$$\mu(\{x \in B_1 \,|\, u = 0\}) \geq \epsilon\mu(B_1), \int_{B_1} u_m^2\,dx = 1, \int_{B_1} |Du_m|^2\,dx \longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

Hence, we may assume $u_m \longrightarrow u_0 \in H^1(B_1)$ strongly in $L^2(B_1)$ and weakly in $H^1(B_1)$. Clearly, u_0 is a non-zero constant. Thus,

$$\begin{aligned} 0 = \lim_{m\longrightarrow\infty} \int_{B_1} |u_m - u_0|^2\,dx &\geq \lim_{m\longrightarrow\infty} \int_{\{u_m=0\}} |u_m - u_0|^2\,dx \\ &\geq |u_0|^2 \inf_m \mu(\{u_m = 0\}) > 0, \end{aligned}$$

which is a contradiction. □

If u is some positive weak solution, or more generally a supersolution, and it is bounded uniformly away from zero in a measurable portion of the domain, then we can use the previous two lemmas to prove that u is locally bounded away from zero.

Theorem 3.29 (Density)**.** *Suppose u is a positive supersolution in B_2 with*

$$\mu(\{x \in B_1 \,|\, u \geq 1\}) \geq \epsilon\mu(B_1).$$

Then there exists a constant C depending only on ϵ, n, and Λ/λ such that

$$\inf_{B_{1/2}} u \geq C.$$

Proof. We may assume that $u \geq \delta > 0$. Then let $\delta \longrightarrow 0$. By Lemma 3.16, $v = (\log u)^-$ is a subsolution, bounded by $\log \delta^{-1}$. Then Theorem 3.27 implies

$$\sup_{B_{1/2}} v \leq C\left(\int_{B_1} |v|^2\,dx\right)^{\frac{1}{2}}.$$

Observe that $\mu(\{x \in B_1 \,|\, v = 0\}) = \mu(\{x \in B_1 \,|\, u \geq 1\}) \geq \epsilon\mu(B_1)$. Lemma 3.17 implies

$$\sup_{B_{1/2}} v \leq C\left(\int_{B_1} |Dv|^2\,dx\right)^{\frac{1}{2}}. \tag{3.99}$$

Set $\varphi = \zeta/u$ for $\zeta \in C_0^1(B_2)$ as the test function. Then

$$0 \le \int a^{ij}(x) D_i u \, D_j \left(\frac{\zeta^2}{u} \right) dx = - \int \zeta^2 \frac{a^{ij}(x) D_i u \, D_j u}{u^2} \, dx + 2 \int \frac{\zeta a^{ij}(x) D_i u \, D_j \zeta}{u} \, dx,$$

which implies

$$\int \zeta^2 |D \log u|^2 \, dx \le C \int |D\zeta|^2 \, dx.$$

Thus, for fixed $\zeta \in C_0^1(B_2)$ with $\zeta \equiv 1$ in B_1, we obtain

$$\int_{B_1} |D \log u|^2 \, dx \le C.$$

Combining this with (3.99) yields

$$\sup_{B_{1/2}} v = \sup_{B_{1/2}} (\log u)^- \le C,$$

which implies

$$\inf_{B_{1/2}} u \ge e^{-C} > 0.$$

□

The preceding density theorem will be used to control the oscillation of a weak solution u, which is the key ingredient in deriving its local Hölder continuity.

Theorem 3.30 (Oscillation)**.** *Suppose that u is a bounded solution of $Lu = 0$ in B_2. Then there exists a $\gamma = \gamma(n, \Lambda/\lambda) \in (0, 1)$ such that*

$$osc_{B_{1/2}} u \le \gamma \, osc_{B_1} u. \tag{3.100}$$

Remark 3.15. *Recall the oscillation of f over the set S is given by*

$$osc_S(f) := \sup_{x \in S} f(x) - \inf_{x \in S} f(x).$$

Proof. In fact, local boundedness follows from Theorem 3.27. Set

$$\alpha_1 = \sup_{B_1} u \ \text{ and } \ \beta_1 = \inf_{B_1} u.$$

Consider the solution

$$\frac{u - \beta_1}{\alpha_1 - \beta_1} \ \text{ or } \ \frac{\alpha_1 - u}{\alpha_1 - \beta_1}.$$

Note the following equivalence:

$$u \ge \frac{1}{2}(\alpha_1 + \beta_1) \Longleftrightarrow \frac{u - \beta_1}{\alpha_1 - \beta_1} \ge \frac{1}{2},$$
$$u \le \frac{1}{2}(\alpha_1 + \beta_1) \Longleftrightarrow \frac{\alpha_1 - u}{\alpha_1 - \beta_1} \ge \frac{1}{2}.$$

Case 1: Suppose that

$$\mu\left(\left\{x \in B_1 \,:\, \frac{2(u-\beta_1)}{\alpha_1-\beta_1} \geq 1\right\}\right) \geq \frac{1}{2}\mu(B_1).$$

Applying the density theorem to $\frac{u-\beta_1}{\alpha_1-\beta_1} \geq 0$ in B_1, we get for some constant $C > 1$

$$\inf_{B_{1/2}} \frac{u-\beta_1}{\alpha_1-\beta_1} \geq \frac{1}{C},$$

which implies

$$\inf_{B_{1/2}} u \geq \beta_1 + \frac{1}{C}(\alpha_1 - \beta_1).$$

Case 2: Suppose that

$$\mu\left(\left\{x \in B_1 \,:\, \frac{2(\alpha_1-u)}{\alpha_1-\beta_1} \geq 1\right\}\right) \geq \frac{1}{2}\mu(B_1).$$

Applying the density theorem as before and noting that $\sup_{B_{1/2}} u = \inf_{B_{1/2}} -u$, we obtain

$$\sup_{B_{1/2}} u \leq \alpha_1 - \frac{1}{C}(\alpha_1 - \beta_1).$$

Now set

$$\alpha_2 = \sup_{B_{1/2}} u \ \text{ and } \ \beta_2 = \inf_{B_{1/2}} u,$$

and note that $\beta_2 \geq \beta_1$ and $\alpha_2 \leq \alpha_1$. In both cases, we have

$$\alpha_2 - \beta_2 \leq \left(1 - \frac{1}{C}\right)(\alpha_1 - \beta_1).$$

This is precisely the estimate (3.100) with $\gamma = 1 - 1/C \in (0, 1)$. □

At last, we are now equipped to state and prove De Giorgi's Hölder regularity theorem.

Theorem 3.31 (De Giorgi)**.** *Suppose $Lu = 0$ weakly in B_1. Then there holds*

$$\sup_{B_{1/2}} |u(x)| + \sup_{x,y \in B_{1/2}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C(n, \Lambda/\lambda)\|u\|_{L^2(B_1)},$$

where $\alpha = \alpha(n, \Lambda/\lambda) \in (0, 1)$.

Proof. The first part of the estimate follows from Theorem 3.27; that is,

$$\sup_{B_{1/2}} |u(x)| \leq C(n, \Lambda/\lambda)\|u\|_{L^2(B_1)}.$$

We prove the second part of the estimate. Fix any two distinct points $x, y \in B_{1/2}$, set $r = |x - y|$ and let

$$\omega(r) := osc_{B_r}(u) = \sup_{B_r} u - \inf_{B_r} u.$$

By Theorem 3.30 and rescaling, we obtain that

$$\omega(r/2) \leq \gamma\omega(r).$$

Hence, Lemma 3.14 implies that

$$\omega(r) \leq Cr^{\alpha}\omega(1/2) \text{ for all } 0 < r \leq 1/2,$$

where $\alpha = \alpha(n, \Lambda/\lambda)$ is some number in $(0, 1)$. By Theorem 3.27, we have that

$$\omega(1/2) \leq \sup_{B_{1/2}} |u(x)| \leq C\|u\|_{L^2(B_1)}.$$

Inserting this into the previous estimate yields

$$\omega(r) \leq Cr^{\alpha}\|u\|_{L^2(B_1)},$$

which further implies

$$\sup_{x,y \in B_{1/2}} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq C(n, \Lambda/\lambda)\|u\|_{L^2(B_1)}.$$

This completes the proof.

□

3.7.5 Hölder Regularity: the Weak Harnack Inequality

We now state and prove the weak Harnack inequality. As a result, we derive Moser's Harnack inequality as a special case, and we combine it with our previous local boundedness result to give another proof of the interior Hölder continuity of weak solutions. Then, we also examine applications of the weak Harnack inequality to obtain a Liouville type theorem and a version of the strong maximum principles for weak solutions.

For simplicity, we only consider elliptic equations without lower order terms. Suppose $U \subset \mathbb{R}^n$, $a^{ij} \in L^{\infty}(U)$ satisfies

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } x \in U \text{ and } \xi \in \mathbb{R}^n$$

for some positive constants λ and Λ.

Theorem 3.32 (Weak Harnack inequality)**.** *Let $u \in H^1(U)$ be a non-negative supersolution in U, i.e.,*

$$\int_U a^{ij}(x) D_i u D_j \varphi \, dx \geq \int_U f \varphi \, dx \ \text{ for any non-negative } \varphi \in H_0^1(U). \tag{3.101}$$

Suppose $f \in L^q(U)$ for some $q > n/2$. Then for any $B_R \subset U$, there holds for any $p \in (0, \frac{n}{n-2})$ and any $0 < \theta < \tau < 1$,

$$\inf_{B_{\theta R}} u + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \geq C\Big(\frac{1}{R^n} \int_{B_{\tau R}} u^p \Big)^{\frac{1}{p}}$$

where $C = C(n, \lambda, \Lambda, p, q, \theta, \tau)$ is a positive constant.

The proof of the weak Harnack inequality and the result on the Hölder continuity of weak solutions will make use of the following result, which is a special case of the local boundedness result of Theorem 3.27.

Theorem 3.33 (local boundedness)**.** *Let $u \in H^1(U)$ be a non-negative subsolution in U in the following sense:*

$$\int_U a^{ij}(x) D_i u D_j \varphi \, dx \leq \int_U f \varphi \, dx \ \text{ for any non-negative } \varphi \in H_0^1(U).$$

Suppose $f \in L^q(U)$ for some $q > n/2$. Then there holds for any $B_R \subset U$, any $r \in (0, R)$, and any $p > 0$,

$$\sup_{B_r} u \leq C \Big\{ \frac{1}{(R-r)^{n/p}} \|u^+\|_{L^p(B_R)} + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \Big\}$$

where $C = C(n, \lambda, \Lambda, p, q)$ is a positive constant.

Proof of the weak Harnack inequality. We prove this for $R = 1$.

Step 1: We prove the result for some $p = p_0 > 0$. Set $\bar{u} = u + k > 0$ for some $k > 0$ to be determined below and $v = \bar{u}^{-1}$. First, we derive the equation for $v(x)$. For any non-negative $\varphi \in H_0^1(B_1)$, let the function $\bar{u}^{-2}\varphi$ be the test function in equation (3.101). Then

$$\int_{B_1} a^{ij}(x) D_i u \frac{D_j \varphi}{\bar{u}^2} \, dx - 2 \int_{B_1} a^{ij}(x) D_i u D_j \bar{u} \frac{\varphi}{\bar{u}^3} \, dx \geq \int_{B_1} f \frac{\varphi}{\bar{u}^2} \, dx$$

Note that $D\bar{u} = Du$ and $Dv = -\bar{u}^2 D\bar{u}$. Therefore, we obtain

$$\int_{B_1} a^{ij}(x) D_j v D_i \varphi + \bar{f} v \varphi \, dx \leq 0 \ \text{ where } \ \bar{f} := \frac{f}{\bar{u}}.$$

That is, v is a non-negative subsolution to some homogeneous equation. Choose $k = \|f\|_{L^q(U)}$ if $f \not\equiv 0$. Otherwise, choose arbitrary $k > 0$ and let $k \to 0$. Note $\|\bar{f}\|_{L^q(B_1)} \leq 1$. Thus, Theorem 3.33 implies that for any $\tau \in (\theta, 1)$ and any $p > 0$,

$$\sup_{B_\theta} \bar{u}^{-p} \leq C \int_{B_\tau} \bar{u}^{-p} \, dx,$$

that is, we deduce the desired estimate

$$\inf_{B_\theta} \bar{u} \geq C\Big(\int_{B_\tau} \bar{u}^{-p}\,dx\Big)^{-\frac{1}{p}} = C\Big(\int_{B_\tau} \bar{u}^{-p}\,dx \int_{B_\tau} \bar{u}^{p}\,dx\Big)^{-\frac{1}{p}}\Big(\int_{B_\tau} \bar{u}^{p}\,dx\Big)^{\frac{1}{p}},$$

where $C = C(n, \lambda, \Lambda, p, q, \theta, \tau)$ is a positive constant. The main step here is to prove there exists a $p_0 > 0$ such that

$$\int_{B_\tau} \bar{u}^{-p_0}\,dx \cdot \int_{B_\tau} \bar{u}^{p_0}\,dx \leq C(n, \lambda, \Lambda, p, q, \tau). \tag{3.102}$$

To show this, it suffices to prove the following claim:
For any $\tau < 1$,

$$\int_{B_\tau} e^{p_0|w|}\,dx \leq C(n, \lambda, \Lambda, p, q)\tau^n \text{ or } C(n, \lambda, \Lambda, p, q, \tau) \tag{3.103}$$

where

$$w = \log \bar{u} - \beta \ \text{ with } \ \beta = |B_\tau|^{-1}\int_{B_\tau} \log \bar{u}\,dx,$$

since this claim and the fact that $-p_0|w| \leq \pm p_0 w \leq p_0|w|$ would imply that

$$\begin{aligned}\int_{B_r} \bar{u}^{-p_0}\,dx\Big(\int_{B_r} \bar{u}^{p_0}\,dx\Big) &= \int_{B_r} e^{-p_0\beta} e^{\log \bar{u}^{p_0}}\,dx \int_{B_r} e^{p_0\beta} e^{\log \bar{u}^{-p_0}}\,dx \\ &= \int_{B_r} e^{-p_0 w}\,dx \int_{B_r} e^{p_0 w}\,dx \leq C(n, \lambda, \Lambda, p, q, \tau).\end{aligned}$$

To prove estimate (3.103), we notice that it follows directly from the John-Nirenberg lemma, i.e., Lemma 3.2, provided that we show $w \in BMO$, i.e.,

$$\frac{1}{r^n}\int_{B_r} |w - w_{y,r}|\,dx \leq C.$$

We first derive the equation for w. As before, consider $\bar{u}^{-1}\varphi$ to be the test function in (3.101) and assume that φ is non-negative with $\varphi \in L^\infty(B_1) \cap H_0^1(B_1)$. By direct calculations and the fact that $Dw = \bar{u}^{-1}Du$, we get that

$$\int_{B_1} a^{ij}(x) D_i w D_j(w\varphi)\,dx \leq \int_{B_1} a^{ij}(x) D_i w D_j \varphi\,dx + \int_{B_1} -\bar{f}\varphi\,dx \tag{3.104}$$

for any non-negative $\varphi \in L^\infty(B_1) \cap H_0^1(B_1)$. Replace φ by φ^2 in (3.104). Then Hölder's inequality yields

$$\int_{B_1} |Dw|^2\varphi^2\,dx \leq C\Big(\int_{B_1} |D\varphi|^2\,dx + \int_{B_1} |\bar{f}|\varphi^2\,dx\Big). \tag{3.105}$$

Furthermore, Hölder's inequality and the Sobolev embedding imply

$$\int_{B_1} |\bar{f}|\varphi^2\,dx \leq \|\bar{f}\|_{L^{n/2}(B_1)}\|\varphi\|^2_{L^{\frac{2n}{n-2}}(B_1)} \leq C(n, q)\|D\varphi\|^2_{L^2(B_1)}.$$

Hence,

$$\int_{B_1} |Dw|^2 \varphi^2 \, dx \le C(n, q, \lambda, \Lambda) \int_{B_1} |D\varphi|^2 \, dx. \tag{3.106}$$

Here, we can choose φ to be in $C_0^1(B_1)$. Moreover, for any $B_{2r}(y) \subset B_1$, we can choose φ with $supp\, \varphi \subset B_{2r}(y)$, $\varphi \equiv 1$ in $B_r(y)$, and $|D\varphi| \le \frac{2}{r}$. Then

$$\int_{B_r(y)} |Dw|^2 \, dx \le Cr^{n-2}.$$

Hence, Poincare's inequality yields

$$\frac{1}{r^n} \int_{B_r(y)} |w - w_{y,r}| \, dx \le \frac{1}{r^{n/2}} \Big(\int_{B_r(y)} |w - w_{y,r}|^2 \, dx \Big)^{\frac{1}{2}} \le \frac{1}{r^{n/2}} \Big(r^2 \int_{B_r(y)} |Dw|^2 \, dx \Big)^{\frac{1}{2}} \le C.$$

That is, $w \in BMO$ and this proves the claim.

Step 2: We now verify the result for any $p \in (0, \frac{n}{n-2})$, but we only sketch the main steps as it is similar to the proof of Theorem 3.27. It suffices to prove the following claim. Namely, by the existence of p_0 from Step 1, Moser's iteration scheme yields, for any $0 < r_1 < r_2 < 1$ and $0 < p_2 < p_1 < \frac{n}{n-2}$,

$$\Big(\int_{B_{r_1}} \bar{u}^{p_1} \, dx \Big)^{\frac{1}{p_1}} \le C(n, q, \lambda, \Lambda, r_1, r_2, p_1, p_2) \Big(\int_{B_{r_2}} \bar{u}^{p_2} \, dx \Big)^{\frac{1}{p_2}}. \tag{3.107}$$

To start, we take $\varphi = \bar{u}^{-\beta-1}\eta^2$ for $\beta \in (0, 1)$ as the test function in (3.101). Then, we can establish that

$$\int_{B_1} |D\bar{u}|^2 \bar{u}^{-\beta-1}\eta^2 \, dx \le C \Big\{ \frac{1}{\beta^2} \int_{B_1} |D\eta|^2 \bar{u}^{1-\beta} \, dx + \frac{1}{\beta} \int_{B_1} \frac{|f|}{k} \eta^2 \bar{u}^{1-\beta} \, dx \Big\}.$$

Set $\gamma = 1 - \beta \in (0, 1)$ and $w = \bar{u}^{\gamma/2}$. Then we have

$$\int |Dw|^2 \eta^2 \, dx \le \frac{C}{(1-\gamma)^\alpha} \int w^2 (|D\eta|^2 + \eta^2) \, dx$$

or

$$\int |D(w\eta)|^2 dx \le \frac{C}{(1-\gamma)^\alpha} \int w^2 (|D\eta|^2 + \eta^2) \, dx$$

for some positive $\alpha > 0$. By the Sobolev embedding and a proper choice of a cutoff function with $\chi = n/(n-2)$, we obtain for any $\gamma \in (0, 1)$ and $0 < r < R < 1$,

$$\Big(\int_{B_r} w^{2\chi} \, dx \Big)^{1/\chi} \le \frac{C}{(1-\gamma)^\alpha} \frac{1}{(R-r)^2} \int_{B_R} w^2 \, dx,$$

or

$$\begin{aligned} \Big(\int_{B_r} \bar{u}^{\gamma\chi} \, dx \Big)^{1/\gamma\chi} &\le \Big(\frac{C}{(1-\gamma)^\alpha} \frac{1}{(R-r)^2} \Big)^{1/\gamma} \Big(\int_{B_R} \bar{u}^\gamma \, dx \Big)^{1/\gamma} \\ &\le \Big(\frac{C(1+\gamma)^{1+\sigma}}{R-r} \Big)^{2/\gamma} \Big(\int_{B_R} \bar{u}^\gamma \, dx \Big)^{1/\gamma} \end{aligned} \tag{3.108}$$

for some $\sigma > 0$. We may iterate this last estimate finitely-many times to get (3.107). □

A special case of the weak Harnack inequality is Moser's version.

Theorem 3.34 (Moser's Harnack inequality)**.** *Let $u \in H^1(U)$ be a non-negative solution in U, i.e.,*

$$\int_U a^{ij}(x) D_i u D_j \varphi \, dx = \int_U f\varphi \, dx \ \text{ for any } \varphi \in H_0^1(U).$$

Suppose $f \in L^q(U)$ for some $q > n/2$. Then there holds for any $B_R \subset U$,

$$\max_{B_{R/2}} u \leq C\Big(\min_{B_{R/2}} u + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \Big)$$

where $C = C(n, \lambda, \Lambda, q)$ is a positive constant.

The proof of Moser's version of the Harnack inequality follows from the weak version and Lemma 3.13.

Proof of Moser's Harnack Inequality. Define $\Phi(p, r)$ by

$$\Phi(p, r) := \Big(\int_{B_r} \bar{u}^p \, dx \Big)^{1/p}.$$

Then (3.108) implies the estimate

$$\Phi(\chi\gamma, r) \leq \Big(\frac{C(1+\gamma)^{\sigma+1}}{R-r} \Big)^{2/\gamma} \Phi(\gamma, R). \tag{3.109}$$

Set for $m = 0, 1, 2, 3, \ldots,$

$$\gamma = \gamma_m = \chi^m p \ \text{ and } \ r_m = 1/2 + 2^{-(m+1)}.$$

Then, by iterating estimate (3.109), we get

$$\Phi(\chi^m \gamma, 1/2) \leq (C\chi)^{2(1+\sigma)\sum m\chi^{-m}} \Phi(p, 1).$$

By sending $m \longrightarrow \infty$ here and applying Lemma 3.13, we arrive at

$$\sup_{B_{1/2}} \bar{u} \leq C\Phi(p, 1).$$

The desired estimate follows from this and the weak Harnack inequality. □

Now, our goal is to establish the Hölder continuity of weak solutions using the local boundedness result and Moser's Harnack inequality.

Corollary 3.3 (Hölder continuity)**.** *Let $u \in H^1(U)$ be a solution of the equation in U:*

$$\int_U a^{ij}(x) D_i u D_j \varphi \, dx = \int_U f\varphi \, dx \ \text{ for any } \ \varphi \in H_0^1(U).$$

Suppose $f \in L^q(U)$ for some $q > n/2$. Then $u \in C^\alpha(U)$ for some $\alpha \in (0,1)$ depending only on n, q, λ and Λ. Moreover, there holds for any $B_R \subset U$

$$|u(x) - u(y)| \le C\Big(\frac{|x-y|}{R}\Big)^\alpha \Big\{\Big(\frac{1}{R^n}\int_{B_R} u^2 \, dx\Big)^{\frac{1}{2}} + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)}\Big\}$$

for any $x, y \in B_{R/2}$ where $C = C(n, \lambda, \Lambda, q)$ is a positive constant.

Proof. We prove the estimate for the case $R = 1$. Set for $r \in (0,1)$

$$M(r) = \max_{B_r} u \ \text{ and } \ m(r) = \min_{B_r} u.$$

Then $M(r) < \infty$ and $m(r) > -\infty$. It suffices to prove for any $r < 1/2$,

$$\omega(r) := M(r) - m(r) \le Cr^\alpha \Big\{\Big(\int_{B_1} u^2 \, dx\Big)^{\frac{1}{2}} + \|f\|_{L^q(B_1)}\Big\}. \tag{3.110}$$

Set $\delta = 2 - n/q$ and apply Theorem 3.34 to $M(r) - u \ge 0$ in B_r to get

$$\sup_{B_{r/2}} (M(r) - u) \le C\Big\{\inf_{B_{r/2}} (M(r) - u) + r^\delta \|f\|_{L^q(B_r)}\Big\}.$$

Combining this with the definitions of the supremum and infimum, we get

$$\begin{aligned} \inf_{B_{r/2}} (M(r) - u) &\le \sup_{B_{r/2}} (M(r) - u) \\ &\le C\Big\{\inf_{B_{r/2}} (M(r) - u) + r^\delta \|f\|_{L^q(B_r)}\Big\} \le C\Big\{\sup_{B_{r/2}} (M(r) - u) + r^\delta \|f\|_{L^q(B_r)}\Big\}. \end{aligned}$$

Hence,

$$M(r) - m(r/2) \le C\Big\{(M(r) - M(r/2)) + r^\delta \|f\|_{L^q(B_r)}\Big\}. \tag{3.111}$$

Likewise, applying the same argument to $u - m(r) \ge 0$ in B_r, we get

$$M(r/2) - m(r) \le C\Big\{(m(r/2) - m(r)) + r^\delta \|f\|_{L^q(B_r)}\Big\}. \tag{3.112}$$

Adding (3.111) and (3.112) together yields

$$\omega(r) + \omega(r/2) \le C\Big\{(\omega(r) - \omega(r/2)) + r^\delta \|f\|_{L^q(B_r)}\Big\}$$

or

$$\omega(r/2) \le \gamma \omega(r) + Cr^\delta \|f\|_{L^q(B_r)}$$

for some $\gamma = (C-1)/(C+1) < 1$.

Apply Lemma 3.14 with μ is chosen such that $\alpha = (1-\mu)\log\gamma/\log\tau < \mu\delta$. Then

$$\omega(\rho) \le C\rho^{\alpha}\{\omega(1/2) + \|f\|_{L^q(B_1)}\} \text{ for any } \rho \in (0, 1/2]. \tag{3.113}$$

On the other hand, Theorem 3.33 implies

$$\omega(1/2) \le C\Big\{\Big(\int_{B_1} u^2\,dx\Big)^{\frac{1}{2}} + \|f\|_{L^q(B_1)}\Big\}$$

and inserting this into (3.113) completes the proof of the corollary. □

3.7.6 Further Applications of the Weak Harnack Inequality

A Liouville theorem

First, we point out an application of Lemma 3.14. Namely, we can derive the following Liouville theorem.

Theorem 3.35. *Suppose $u \in H^1(U)$ is a solution to the homogeneous equation in $\mathbb{R}^n$:*

$$\int_{\mathbb{R}^n} a^{ij}(x)D_iuD_j\varphi\,dx = 0 \text{ for any } \varphi \in H^1_0(\mathbb{R}^n).$$

If u is bounded, then u is constant.

Proof. From the previous corollary, we showed that there exists a $\gamma < 1$ such that

$$\omega(r) \le \gamma\omega(2r).$$

By iteration, we obtain

$$\omega(r) \le \gamma^k\omega(2^kr) \to 0 \text{ as } k \to \infty$$

since $\omega(2^kr) \le C$ if u is bounded. Hence, for any $r > 0$,

$$\omega(r) = 0.$$

Thus, $u \equiv$ *constant*. □

Maximum principles for weak solutions

An application of the weak Harnack inequality is the strong maximum principle adapted for weak solutions. However, we introduce some necessary definitions and consider the weak maximum principle for weak solutions. We say that $u \in H^1(U)$ satisfies $u \le 0$ on ∂U if its positive part $u^+ = \max\{u, 0\}$ belongs to $H^1_0(U)$. Of course if u is continuous in a neighborhood of ∂U then u satisfies $u \le 0$ on ∂U if the inequality holds in the classical pointwise sense. Likewise, we say $u \geq 0$ on ∂U if $-u \leq 0$ on ∂U; and $u \leq v \in H^1(U)$ on ∂U if $u - v \leq 0$ on ∂U. As usual, we take

$$Lu = -D_i(a^{ij}(x)D_j u)$$

and solutions, supersolutions, and subsolutions associated with this elliptic operator are understood in the distributional sense.

Theorem 3.36 (Weak Maximum Principle for Weak Solutions)**.** *Let $u \in H^1(U)$.*

(a) If $Lu \leq 0$ in U, then $\sup_U u \leq \sup_{\partial U} u^+$.

(b) If $Lu \geq 0$ in U, then $\inf_U u \geq \inf_{\partial U} u^-$.

Proof. Since $Lu \leq 0$ in U in the distribution sense, we write

$$\int_U a^{ij}(x) D_j u D_i v \, dx \leq 0$$

for all non-negative $v \in H^1_0(U)$. If we set $\ell = \sup_{\partial U} u^+$ and take $v = \max\{u - \ell, 0\}$, then $v \in H^1_0(U)$, $Dv = Du$ if $u - \ell > 0$ and $Dv = 0$ if $u - \ell \leq 0$. We proceed by contradiction. That is, assume $v > 0$ or $u > \ell$ in some subset $B \subset\subset U$ with $\mu(B) > 0$; otherwise, if $v \equiv 0$ then we would be done. Clearly, $Dv = Du$ within B; but the positivity of $(a^{ij}(x))$ and the uniform ellipticity condition imply that

$$\int_B |Dv|^2 \, dx \leq 0,$$

and we get that v, and therefore u, is constant in a subset of U with positive measure. At the same time, a basic result guarantees $Du = 0$ *a.e.* in this subset and we deduce a contradiction. This completes the proof for part (a). Part (b) follows along a similar argument; namely, we can apply the previous proof to $-Lu \leq 0$ and the fact that $\inf_D u = -\sup_D(-u)$.

□

From this, we immediately deduce a uniqueness result.

Corollary 3.4. *Let $u \in H^1_0(U)$ satisfy $Lu = 0$ in U. Then $u = 0$ in U.*

We are now ready to introduce the strong version of the maximum principle adapted for weak solutions. Unlike the weak maximum principle above, we are only assuming the weak solution belongs to $H^1(U)$. We do not assume the solution vanishes at the boundary in the trace sense, i.e., it does not necessarily belong to $H^1_0(U)$. The Harnack inequality plays an essential role in its proof.

Theorem 3.37 (Strong Maximum Principle for Weak Solutions)**.** *Let U be a bounded and open subset and let $u \in H^1(U)$ satisfy $Lu \leq 0$ in U. Then, if for some ball $B \subset\subset U$ we have*

$$\sup_B u = \sup_U u \geq 0, \tag{3.114}$$

the function u must be constant in U.

Proof. Denote $B = B_R(y)$ and without loss of generality, we can assume that $B_{4R}(y) \subset U$. Now let $M = \sup_U u$ and then apply the weak Harnack inequality (see Theorem 3.32) with $p = 1$ to the supersolution $v = M - u$. Namely, we use the following dilated version of the weak Harnack inequality with $p = 1$:

$$R^{-n}\|v\|_{L^1(B_{2R}(y))} \leq C \inf_{B_R(y)} v.$$

Hence,

$$R^{-n} \int_{B_{2R}} (M - u)\, dx \leq C \inf_B (M - u) = 0$$

and so $u \equiv M$ in B_{2R}. Therefore, supremum of u is attained for a larger ball in U. We can then show $u \equiv M$ in U by a simple covering argument. □

Remark 3.16. *Likewise, we have an analogous result which states the solution to $Lu \geq 0$ in U is constant whenever it attains an interior minimum.*

CHAPTER 4

Viscosity Solutions and Fully Nonlinear Equations

4.1 Introduction

This chapter introduces a very weak concept of solution for second-order elliptic equations called viscosity solutions. To simplify our presentation, the results given here are for equations involving linear elliptic operators without lower order terms, but they can certainly be extended to fully nonlinear elliptic equations of the type

$$F(D^2u, u, x) = f(x) \text{ in } U,$$

where $F : \mathbb{R}^{n\times n} \times \mathbb{R} \times \mathbb{R}^n$ is usually a monotone and convex mapping possibly nonlinear in D^2u and u. For a nice introductory treatment of this topic, we refer the reader to Caffarelli and Cabré [4].

The advantage of considering the notion of viscosity solution is it allows us to consider elliptic equations in non-divergence form, and it extends the notion of classical solutions. Another advantage is that viscosity solutions are stable under local uniform convergence in both u and F and because existence and uniqueness results for such solutions can be obtained under far more general conditions. In fact, in the definition given below, notice that we can make sense of such solutions without resorting to differentiating the equations directly. This was a major obstacle in extending elliptic theory to equations having non-divergence form, since the usual procedure of integrating by parts and treating equations in the distribution sense was not generally possible, or the usual notions of solution was not always guaranteed to exist in this context. Thus, finding a successful framework that circumvents this obstacle was a tremendous breakthrough in the modern theory of elliptic partial differential equations.

The results we establish below should be reminiscent of those for elliptic equations in divergence form studied earlier, however, we obtain the results via perturbation methods relying heavily on approximation and density arguments. More precisely, we shall give a concise introduction, develop the Alexandroff maximum principle along with a Harnack inequality for viscosity solutions. Then we use these to develop the interior Schauder and $W^{2,p}$ regularity estimates for viscosity solutions. Global versions of these regularity results without proof are also provided at the end of the chapter.

Let U be a bounded and connected domain in $\mathbb{R}^n$ and (a^{ij}) is of class $C(U)$ and satisfies

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for any $x \in U$ and any $\xi \in \mathbb{R}^n$. We consider the operator L in U defined by

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u \text{ for } u \in C^2(U). \tag{4.1}$$

Throughout, we shall assume that f belongs to $C(U)$.

Definition 4.1. *The function $u \in C(U)$ is said to be a* **viscosity supersolution** *(respectively* **viscosity subsolution***) of the equation*

$$Lu = f \text{ in } U \tag{4.2}$$

if for any $x_0 \in U$ and any function $\varphi \in C^2(U)$ such that $u - \varphi$ has a local minimum (respectively, local maximum) at x_0 there holds

$$L\varphi(x_0) \geq f(x_0) \text{ (respectively, } L\varphi(x_0) \leq f(x_0)).$$

The following definition of solution should be compared with the result of Theorem 1.10.

Definition 4.2. *We say $u \in C(U)$ is a* **viscosity solution** *of equation* (4.1) *if it is both a viscosity subsolution and a viscosity supersolution.*

Remark 4.1. *By density, the C^2 function φ in the above definitions may be replaced by quadratic polynomials.*

Next we look at the class of all solutions to all elliptic equations. First we make the following important observation. Let $e_1, e_2, \ldots, e_n$ be the eigenvalues of the Hessian matrix $D^2\varphi(x_0)$ where φ is any C^2 function at $x_0 \in U$. We have the following chain of equivalent estimates:

$$\begin{aligned}\sum_{i,j=1}^{n} a^{ij}(x_0)D_{ij}\varphi(x_0) \leq 0 &\iff \sum_{i=1}^{n} \alpha_i e_i \leq 0 \text{ for } \alpha_i \in [\lambda, \Lambda],\\ &\iff \sum_{e_i>0} \alpha_i e_i + \sum_{e_i<0} \alpha_i e_i \leq 0,\\ &\iff \sum_{e_i>0} \alpha_i e_i \leq \sum_{e_i<0} \alpha_i(-e_i),\end{aligned}$$

where the last line implies
$$\lambda \sum_{e_i>0} \alpha_i e_i \le \Lambda \sum_{e_i<0} \alpha_i(-e_i).$$
Namely, if u is a "supersolution," then the positive eigenvalues of the Hessian matrix $D^2\varphi(x_0)$ are controlled by its negative eigenvalues. This motivates the following definition.

Definition 4.3. *Suppose $f \in C(U)$ and λ and Λ are two positive constants. We define $u \in C(U)$ to belong to $\mathcal{S}^+(\lambda, \Lambda, f)$ if for any $x_0 \in U$ and any function $\varphi \in C^2(U)$ such that $u - \varphi$ has a local minimum at x_0, there holds*

$$\lambda \sum_{e_i(x_0)>0} e_i(x_0) + \Lambda \sum_{e_i<0} e_i(x_0) \ge f(x_0),$$

where $e_1(x_0), e_2(x_0), \ldots, e_n(x_0)$ are eigenvalues of the Hessian matrix $D^2\varphi(x_0)$.

Similarly, we define $u \in C(U)$ to belong to $\mathcal{S}^-(\lambda, \Lambda, f)$ if for any $x_0 \in U$ and any function $\varphi \in C^2(U)$ such that $u - \varphi$ has a local maximum at x_0, there holds

$$\Lambda \sum_{e_i(x_0)>0} e_i(x_0) + \lambda \sum_{e_i<0} e_i(x_0) \le f(x_0).$$

We denote $\mathcal{S}(\lambda, \Lambda, f) = \mathcal{S}^+(\lambda, \Lambda, f) \cap \mathcal{S}^-(\lambda, \Lambda, f)$

Notice that any viscosity supersolution of (4.2) belongs to the class $\mathcal{S}^+(\lambda, \Lambda, f)$. In fact, the class $\mathcal{S}^+(\lambda, \Lambda, f)$ and $\mathcal{S}^-(\lambda, \Lambda, f)$ also include solutions to fully nonlinear equations such as the Pucci equations.

We say the matrix $A = (a^{ij})$ belongs to the class $A_{\lambda,\Lambda}$ with any two constants $\lambda, \Lambda > 0$ if A is symmetric and

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \text{ for } x \in U,\ \xi \in \mathbb{R}^n$$

so that its eigenvalues belong to $[\lambda, \Lambda]$.

Now, for any symmetric matrix $M = (m^{ij})$, we define the Pucci extremal operators:

$$\mathcal{M}^-(M) = \mathcal{M}^-(\lambda, \Lambda, M) = \inf_{A \in A_{\lambda,\Lambda}} a^{ij} m^{ij},$$
$$\mathcal{M}^+(M) = \mathcal{M}^+(\lambda, \Lambda, M) = \sup_{A \in A_{\lambda,\Lambda}} a^{ij} m^{ij}.$$

Then Pucci's equations are given by

$$\mathcal{M}^-(\lambda, \Lambda, M) = f,$$
$$\mathcal{M}^+(\lambda, \Lambda, M) = g,$$

for some functions $f, g \in C(U)$. Indeed, we can show that

$$\mathcal{M}^-(\lambda, \Lambda, M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i,$$
$$\mathcal{M}^+(\lambda, \Lambda, M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where $e_1, e_2, \dots, e_n$ are eigenvalues of M. Hence, $u \in \mathcal{S}^+(\lambda, \Lambda, f)$ if and only if

$$\mathcal{M}^-(\lambda, \Lambda, D^2 u) \leq f$$

in the viscosity sense, i.e., for any $\varphi \in C^2(U)$ such that $u - \varphi$ has a local minimum at $x_0 \in U$ there holds

$$\mathcal{M}^-(\lambda, \Lambda, D^2\varphi(x_0)) \leq f(x_0).$$

An analogous statement holds for $u \in \mathcal{S}^-(\lambda, \Lambda, f)$ and viscosity subsolutions.

By definition of $\mathcal{M}^-$ and $\mathcal{M}^+$, we can check that for any two symmetric matrices M and N,

$$\begin{aligned}\mathcal{M}^-(M) + \mathcal{M}^-(N) \leq \mathcal{M}^-(M+N) &\leq \mathcal{M}^+(M) + \mathcal{M}^-(N) \\ &\leq \mathcal{M}^+(M+N) \leq \mathcal{M}^+(M) + \mathcal{M}^+(N).\end{aligned}$$

This will be an important property we invoke later in establishing the regularity of viscosity solutions. We now establish the **Alexandroff maximum principle** for viscosity solutions, and we may think of it as a replacement of the energy inequality for weak solutions to elliptic equations in divergence form. The Alexandroff maximum principle is sometimes called the **Alexandroff-Bakelman-Pucci** or **ABP** estimate. First, recall that L defined in $\mathbb{R}^n$ is said to be **affine** if

$$L(x) = \ell_0 + \ell(x),$$

where $\ell_0 \in \mathbb{R}$ and ℓ is a linear function. We denote the **convex envelope** of a function v defined in U by

$$\Gamma(v)(x) = \sup_L \{L(x) \,:\, L \leq v \text{ in } U, L \text{ is an affine function}\}$$

for any $x \in U$. The function Γ is indeed a convex function on U, and it is the largest possible affine function below of v. Moreover, the set of points x in which $\Gamma(v)$ touches v from below, i.e., the set $\{v = \Gamma(v)\}$, is called the (lower) contact set of v. The points in the contact set are called contact points. The following lemma is the Alexandroff maximum principle and note that u is not required to be a solution to any elliptic equation. The classical version is stated as follows, which we provide without proof (see Lemma 3.4 in [4] and Section 9.1 in [13] for detailed proofs).

Lemma 4.1. *Suppose u is a $C^{1,1}$ function in B_1 with $u \geq 0$ on ∂B_1. Then*

$$\sup_{B_1} u^- \leq C(n)\Big(\int_{B_1 \cap \{u=\Gamma_u\}} \det(D^2 u)\, dx\Big)^{1/n},$$

where Γ_u is the convex envelope of $-u^- = \min\{u, 0\}$.

The version of this for viscosity solutions is the following, which we will prove with the help of Lemma 4.1.

Theorem 4.1 (Alexandroff Maximum Principle)**.** *Suppose u belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in B_1 with $u \geq 0$ on ∂B_1 for some $f \in C(U)$. Then*

$$\sup_{B_1} u^- \leq C(n, \lambda, \Lambda)\Big(\int_{B_1 \cap \{u=\Gamma_u\}} (f^+)^n\, dx\Big)^{1/n},$$

where Γ_u is the convex envelope of $-u^- = \min\{u, 0\}$.

Proof. The goal is to ultimately apply Lemma 4.1 to the convex envelope $\Gamma_u(x)$. Namely, we need to prove that Γ_u belongs to $C^{1,1}(B_1)$ and at a contact point x_0, we have that

$$f(x_0) \geq 0 \tag{4.3}$$

and

$$L(x) \leq \Gamma_u(x) \leq L(x) + C(n, \lambda, \Lambda)(f(x_0) + \epsilon(x))|x - x_0|^2 \tag{4.4}$$

for some affine function L and any x sufficiently close to x_0 with $\epsilon(x) = o(1)$ as $x \longrightarrow x_0$. Once we prove this claim, clearly (4.4) implies that

$$\det(D^2\Gamma_u)(x) \leq C(n, \lambda, \Lambda) f(x)^n \text{ for a.e. } x \in \{u = \Gamma_u\}.$$

So Lemma 4.1 applied to the function Γ_u implies the result. Therefore, it remains to prove the claim.

Let x_0 be a contact point, i.e., $u(x_0) = \Gamma_u(x_0)$. Without loss of generality, assume $x_0 = 0$. We may also assume, after subtracting a supporting plane at $x_0 = 0$ if necessary, that $u \geq 0$ in B_1 with $u(0) = 0$. Take $h(x) = -\epsilon|x|^2/2$ in B_1. Clearly, $u - h$ has a minimum at 0, and note that the eigenvalues of $D^2h(0)$ is just $-\epsilon$ with multiplicity n. By definition of $\mathcal{S}^+(\lambda, \Lambda, f)$, we have that

$$-n\Lambda\epsilon \leq f(0).$$

We obtain (4.3) after sending $\epsilon \longrightarrow 0$ in the preceding estimate.

Finally, to obtain estimate (4.4), we will prove

$$0 \leq \Gamma_u(x) \leq C(n, \lambda, \Lambda)(f(0) + \epsilon(x))|x|^2 \text{ for } x \in B_1,$$

where $\epsilon(x) = o(1)$ as $x \longrightarrow 0$.

We need to get an estimate for

$$C_r = \frac{1}{r^2} \max_{B_r} \Gamma_u$$

for small $r > 0$. By convexity, Γ_u attains its maximum in the closed ball $\bar{B}_r$ at some point on the boundary, say at $(0, \ldots, 0, r)$. Now the set $\{x \in B_1 : \Gamma_u(x) \leq \Gamma_u(0, \ldots, 0, r)\}$ is convex and contains B_r. Hence,

$$\Gamma_u(x', r) \geq \Gamma_u(0, \ldots, 0, r) = C_r r^2 \text{ for any } x = (x', r) \in B_1.$$

Choose a positive number N to be specified at a later time. Set

$$R_r = \{(x', x_n) : |x'| \leq Nr, |x_n| \leq r\}.$$

We construct a quadratic polynomial that touches u from below in R_r and curves upward very steeply. Set, for some $b > 0$,

$$h(x) = (x_n + r)^2 - b|x'|^2.$$

Then,

(a) for $x_n = -r$, $h \leq 0$;

(b) for $|x'| = Nr$, $h \leq (4 - bN^2)r^2 \leq 0$ if we take $b = 4/N^2$;

(c) for $x_n = r$, $h = 4r^2 - b|x'|^2 \leq 4r^2$.

Therefore, if we take

$$\tilde{h}(x) = \frac{C_r}{4} h(x) = \frac{C_r}{4}\Big((x_n + r)^2 - \frac{4}{N^2}|x'|^2\Big),$$

and since Γ_u is the convex envelope of u, we have $\tilde{h} \leq \Gamma_u \leq u$ on ∂R_r. Moreover, $\tilde{h}(0) = C_r r^2/4 > 0 = \Gamma_u(0) = u(0)$. Then, after lowering $\tilde{h}$ if necessary, we deduce that $u - \tilde{h}$ has a local minimum in the interior of R_r. It is easily checked that the eigenvalues of $D^2\tilde{h}$ are

$$C_r/2, -2C_r/N^2, \ldots, -2C_r/N^2.$$

Hence, by definition of $\mathcal{S}^+(\lambda, \Lambda, f)$, we have that

$$\lambda \frac{C_r}{2} - 2\Lambda(n-1)\frac{C_r}{N^2} \leq \max_{R_r} f.$$

We can now choose N suitably large, which depends only on n, λ and Λ, so that

$$2\Lambda(n-1)/N^2 \leq \lambda/4.$$

Thus, we obtain

$$C_r \leq \frac{4}{\lambda} \max_{R_r} f; \text{ that is, } \max_{B_r} \Gamma_u \leq \frac{4}{\lambda} r^2 \max_{R_r} f.$$

Hence,

$$\Gamma_u(x) \leq \max_{B_r} \Gamma_u \leq \frac{4}{\lambda} \epsilon(r) r^2,$$

where $\epsilon(|x|) = \epsilon(r) = \max_{R_r} f = o(1)$. This completes the proof.

□

Finally, we end this section with a basic result as a consequence of the Calderon-Zygmund decompostion. We will need this result when establishing the Harnack inequality and the regularity theory for viscosity solutions. Here we work in dyadic cubes rather than balls. Q denotes such a dyadic cube after refinement of a given Euclidean domain. We often use $Q_\ell(x_0)$ to denote a dyadic cube centered at $x_0 \in \mathbb{R}^n$ with side length ℓ. Sometimes we omit x_0 if $x_0 = 0$, i.e., $Q_\ell(0) = Q_\ell$.

Lemma 4.2. *Suppose the measurable sets $A \subset B \subset Q_1$ have the following properties.*

(a) $|A| < \delta$ for some $\delta \in (0, 1)$;

(b) for any dyadic cube Q, $|A \cap Q| \geq \delta |Q|$ implies $\tilde{Q} \subset B$ for the predecessor $\tilde{Q}$ of Q.

Then $|A| \leq \delta |B|$.

4.2 A Harnack Inequality

Theorem 4.2 (Harnack inequality)**.** *Suppose u belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in B_1 with $u \geq 0$ in B_1 for some $f \in C(B_1)$. Then*

$$\sup_{B_{1/2}} u \leq C\Big(\inf_{B_{1/2}} u + \|f\|_{L^n(B_1)} \Big) \tag{4.5}$$

where C is a positive constant depending only on n, λ and Λ.

As we have encountered already, Harnack type inequalities imply the interior Hölder regularity of solutions. Thus, we have the following result whose proof we omit but follows similarly to that of Corollary 3.3.

Corollary 4.1. *Suppose u belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in B_1 for some $f \in C(B_1)$. Then $u \in C^\alpha(B_1)$ for some $\alpha \in (0, 1)$ depending only on $n, \lambda,$ and Λ. In particular,*

$$|u(x) - u(y)| \leq C|x - y|^\alpha \Big(\sup_{B_1} |u| + \|f\|_{L^n(B_1)} \Big) \text{ for any } x, y \in B_{1/2}.$$

The main ingredient in proving the Harnack inequality is the following result.

Proposition 4.1. *Suppose u belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in $Q_{4\sqrt{n}}$ with $u \geq 0$ in $Q_{4\sqrt{n}}$ for some $f \in C(Q_{4\sqrt{n}})$. Then there exist two positive constants ϵ_0 and C, depending only on n, λ, and Λ, such that if*

$$\inf_{Q_{1/4}} u \leq 1 \ \textit{ and } \ \|f\|_{L^n(Q_{4\sqrt{n}})} \leq \epsilon_0,$$

then

$$\sup_{Q_{1/4}} u \leq C.$$

To see how Theorem 4.2 follows from this, consider the function

$$u_\delta = \frac{u}{\inf_{Q_{1/4}} u + \delta + \epsilon_0^{-1}\|f\|_{L^n(Q_{4\sqrt{n}})}} \ (\delta > 0),$$

provided that $u \in \mathcal{S}(\lambda, \Lambda, f)$ in $Q_{4\sqrt{n}}$ with $u \geq 0$ in $Q_{4\sqrt{n}}$. Applying Proposition 4.1 to u_δ then sending $\delta \longrightarrow 0$, we get

$$\sup_{Q_{1/4}} u \leq C(\inf_{Q_{1/4}} u + \|f\|_{L^n(Q_{4\sqrt{n}})}).$$

Then estimate (4.5) follows from a standard covering argument.

Lemma 4.3. *Suppose u belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ for some $f \in C(B_{2\sqrt{n}})$. Then there exist constants $\epsilon_0 > 0$, $\mu \in (0, 1)$, and $M > 1$, depending only on n, λ, and Λ, such that if*

$$u \geq 0 \ \textit{ in } \ B_{2\sqrt{n}}, \ \inf_{Q_3} u \leq 1 \ \textit{ and } \ \|f\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0, \tag{4.6}$$

then

$$|\{u \leq M\} \cap Q_1| > \mu.$$

Proof. The idea here to localize where the contact set occurs by choosing suitable functions. Namely, we construct a function g that is "very concave" outside Q_1 so that if we "correct" u by g, the contact set is in Q_1. First note that $B_{1/4} \subset B_{1/2} \subset Q_1 \subset Q_3 \subset B_{2\sqrt{n}}$. Define g in $B_{2\sqrt{n}}$ by

$$g(x) = -M(1 - |x|^2/4n)^\beta$$

for some $\beta > 0$ to be specified later and some $M > 0$. We choose M with respect to β so that

$$g \equiv 0 \ \text{ on } \ \partial B_{2\sqrt{n}}, \ \text{ and } \ g \leq -2 \ \text{ in } \ Q_3. \tag{4.7}$$

Set $w = u + g$ in $B_{2\sqrt{n}}$. We shall prove that w, in particular g, belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}} \backslash Q_1$ provided we choose β large enough. Suppose φ is a quadratic polynomial such that $w - \varphi$ has a local minimum at $x_0 \in B_{2\sqrt{n}}$. Then $u - (\varphi - g)$ has a local minimum at x_0 as well. By definitions of $\mathcal{S}^+(\lambda, \Lambda, f)$ and the Pucci extremal operator $\mathcal{M}^-$,

$$\mathcal{M}^-(\lambda, \Lambda, D^2\varphi(x_0) - D^2g(x_0)) \leq f(x_0),$$

or

$$\mathcal{M}^-(\lambda,\Lambda,D^2\varphi(x_0))+\mathcal{M}^-(\lambda,\Lambda,-D^2g(x_0))\le f(x_0).$$

Therefore, to show g belongs to $\mathcal{S}^+(\lambda,\Lambda,f)$ in $B_{2\sqrt{n}}\backslash Q_1$, it remains to show $\mathcal{M}^-(\lambda,\Lambda,-D^2g(x_0))$ is non-negative. Well, the Hessian matrix of g is given by

$$D_{ij}g(x)=(M\beta/2n)(1-|x|^2/4n)^{\beta-1}\delta_{ij}-[M\beta(\beta-1)/(2n)^2](1-|x|^2/4n)^{\beta-2}x_ix_j.$$

Choose $x=(|x|,0,0,\dots,0)$, then the eigenvalues of $-D^2g(x)$ are given by

$$\begin{aligned}e^+&=(M\beta/2n)(1-|x|^2/4n)^{\beta-2}((2\beta-1)|x|^2/4n-1)\text{ with multiplicity }1,\\ e^-&=-(M\beta/2n)(1-|x|^2/4n)^{\beta-2}\text{ with multiplicity }n-1.\end{aligned}$$

Now choose $\beta>0$ large enough so that $e^+>0$ and $e^-<0$ for $|x|\ge 1/4$. Thus, for $|x|\ge 1/4$, we have

$$\begin{aligned}\mathcal{M}^-(\lambda,\Lambda,-D^2g(x))&=\lambda e^+(x)+(n-1)\Lambda e^-(x)\\ &=\frac{M\beta}{2n}(1-|x|^2/4n)^{\beta-2}\Big[\lambda\Big(\frac{2\beta-1}{4n}|x|^2-1\Big)-(n-1)\Lambda(1-|x|^2/4n)\Big]\\ &\ge 0.\end{aligned}$$

In fact, we have actually proved that

$$w\in\mathcal{S}^+(\lambda,\Lambda,f+\eta)\ \text{ in }\ B_{2\sqrt{n}}$$

for some $\eta\in C_0^\infty(Q_1)$ and $supp(\eta)\subseteq[0,C(n\lambda,\Lambda)]$. We may apply the Alexandroff maximum principle (Theorem 4.1) to w in $B_{2\sqrt{n}}$. Also note that $\inf_{Q_3}w\le -1$ and $w\ge 0$ on $\partial B_{2\sqrt{n}}$ due to (4.6) and (4.7). Thus,

$$\begin{aligned}1&\le C\Big(\int_{B_{2\sqrt{n}}\cap\{w=\Gamma_w\}}(|f|+\eta)^n\,dx\Big)^{1/n}\\ &\le C\|f\|_{L^n(B_{2\sqrt{n}})}+C|\{w=\Gamma_w\}\cap Q_1|^{1/n}.\end{aligned}$$

Choosing ϵ_0 small enough, we get

$$(1/2)\le C|\{w=\Gamma_w\}\cap Q_1|^{1/n}\le C|\{u\le M\}\cap Q_1|^{1/n}$$

since $w(x)=\Gamma_w(x)$ implies $w(x)\le 0$ and thus $u(x)\le -g(x)\le M$. This completes the proof.

□

Next we derive the power decay property of the distribution function of u.

Lemma 4.4. *Let u belong to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ for some $f \in C(B_{2\sqrt{n}})$. Then there exist positive constants ϵ_0, ϵ and C, depending only on n, λ, and Λ, such that if*

$$u \geq 0 \ \ in \ \ B_{2\sqrt{n}}, \inf_{Q_3} u \leq 1 \ \ and \ \ \|f\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0, \tag{4.8}$$

then

$$|\{u \geq t\} \cap Q_1| \leq Ct^{-\epsilon} \ \ for \ \ t > 0.$$

Proof. Under the assumptions (4.8), we claim

$$|\{u > M^k\} \cap Q_1| \leq (1-\mu)^k \ \text{ for } \ k = 1, 2, \ldots, \tag{4.9}$$

where M and μ are the same parameters from Lemma 4.3. We proceed by induction. Indeed, for $k = 1$, (4.9) is just Lemma 4.3. So assume (4.9) holds for $k-1$. Set $A = \{u > M^k\} \cap Q_1$ and $B = \{u > M^{k-1}\} \cap Q_1$. We claim that

$$|A| \leq (1-\mu)|B| \tag{4.10}$$

We do so by using Lemma 4.2. Clearly, $A \subset B \subset Q_1$ and $|A| \leq |\{u > M\} \cap Q_1| \leq 1 - \mu$ by Lemma 4.3. We claim that if $Q = Q_r(x_0)$ is a cube in Q_1 such that

$$|A \cap B| > (1-\mu)|Q|, \tag{4.11}$$

then $\tilde{Q} \cap Q_1 \subset B$ for $\tilde{Q} = Q_{3r}(x_0)$. We prove this by contradiction. Consider the transformation $x = x_0 + ry$ for $y \in Q_1$ and $x \in Q = Q_r(x_0)$, and the function

$$\tilde{u}(y) = M^{-(k-1)} u(x).$$

Then $\tilde{u} \geq 0$ in $B_{2\sqrt{n}}$ and $\inf_{Q_3} \tilde{u} \leq 1$. It is easy to check that $\tilde{u} \in \mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ with $\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0$. In fact,

$$\tilde{f}(y) = \frac{r^2}{M^{k-1}} f(x) \ \text{ for } \ y \in B_{2\sqrt{n}}.$$

Hence,

$$\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \frac{r}{M^{k-1}} \|f\|_{L^n(B_{2\sqrt{n}})} \leq \|f\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0.$$

Therefore, $\tilde{u}$ satisfies (4.8). Thus, Lemma 4.3 applied to $\tilde{u}$ implies

$$\mu < |\{\tilde{u}(y) \leq M\} \cap Q_1| = r^{-n} |\{u(x) \leq M^k\} \cap Q|.$$

Hence, $|Q \cap A^c| > \mu|Q|$, but this contradicts with (4.11). Applying Lemma 4.2 yields (4.10). □

Proof of Proposition 4.1. We show there exist two constants $\theta > 1$ and $M_0 \gg 1$, depending only on n, λ, and Λ, such that if $u(x_0) = P > M_0$ for some $x_0 \in B_{1/4}$ there exists a sequence $\{x_k\} \subset B_{1/2}$ such that

$$u(x_k) \geq \theta^k P \text{ for } k = 0, 1, 2, \ldots.$$

This contradicts with the boundedness of u and thus $\sup_{B_{1/4}} u \leq M_0$.

Suppose $u(x_0) = P > M_0$ for some $x_0 \in B_{1/4}$. We will determine M_0 and θ in the process. Consider a cube $Q_r(x_0)$ centered at x_0 with side length r, which will be specified below. We want to find a point $x_1 \in Q_{4\sqrt{n}r}(x_0)$ such that $u(x_1) \geq \theta P$. To do so, we choose r such that $\{u > P/2\}$ covers less than half of $Q_r(x_0)$. This can be done using the power decay of the distribution function of u (see Lemma 4.4). Namely, since $\inf_{Q_3} u \leq \inf_{Q_{1/4}} u \leq 1$, Lemma 4.4 implies

$$|\{u > P/2\} \cap Q_1| \leq C(P/2)^{-\epsilon}.$$

We choose r such that $r^n/2 \geq C(P/2)^{-\epsilon}$ and $r \leq 1/4$. Hence, we have, for such r, $Q_r(x_0) \subset Q_1$ and

$$\frac{1}{|Q_r(x_0)|}|\{u > P/2\} \cap Q_r(x_0)| \leq 1/2. \tag{4.12}$$

Next we show that for $\theta > 1$, with $\theta - 1$ small, $u \geq \theta P$ at some point in $Q_{4\sqrt{n}r}(x_0)$. We proceed by contradiction. That is, assume $u \leq \theta P$ in $Q_{4\sqrt{n}r}(x_0)$. Consider the transformation

$$x = r_0 + ry \text{ for } Q_{4\sqrt{n}} \text{ and } x \in Q_{4\sqrt{n}r}(x_0)$$

and the function

$$\tilde{u}(y) = \frac{\theta P - u(x)}{(\theta - 1)P}.$$

Clearly, $\tilde{u} \geq 0$ in $B_{2\sqrt{n}}$ and $\tilde{u}(0) = 1$, and thus $\inf_{Q_3} \tilde{u} \leq 1$. It follows that $\tilde{u}$ belongs to $\mathcal{S}^+(\lambda, \Lambda, \tilde{f})$ in $B_{2\sqrt{n}}$ with $\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0$. Indeed, we have

$$\tilde{f}(y) = -\frac{r^2}{(\theta - 1)P} f(x) \text{ for } y \in B_{2\sqrt{n}}$$

and so

$$\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \frac{r}{(\theta - 1)P}\|f\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0$$

provided we choose P so that $r \leq (\theta - 1)P$. Applying Lemma 4.3 to $\tilde{u}$ and noting that $u(x) \leq P/2 \iff \tilde{u}(y) \geq (\theta - 1/2)/(\theta - 1) \gg 1$ provided that θ is close to 1, we get

$$\begin{aligned}\frac{1}{|Q_r(x_0)|}|\{u \leq P/2\} \cap Q_r(x_0)| &= |\tilde{u} \geq (\theta - 1/2)/(\theta - 1)\} \cap Q_1| \\ &\leq C((\theta - 1/2)/(\theta - 1))^{-\epsilon} < 1/2.\end{aligned}$$

This contradicts with (4.12). Hence, we deduce the existence of a $\theta = \theta(n, \lambda, \Lambda) > 1$ such that if

$$u(x_0) = P \text{ for some } x_0 \in B_{1/4},$$

then

$$u(x_1) \geq \theta P \ \text{ for some } \ x_1 \in Q_{4\sqrt{n}r}(x_0) \subset B_{2nr}(x_0)$$

provided that

$$C(n, \lambda, \Lambda) P^{-\epsilon/n} \leq r \leq (\theta - 1)P.$$

Specifically, we need to choose P such that $P \geq (C/(\theta-1))^{n/(n+\epsilon)}$ and then take $r = CP^{-\epsilon/n}$.

Iterating the previous result yields a sequence $\{x_k\}$ such that of for any $k = 1, 2, 3, \ldots,$

$$u(x_k) \geq \theta^k P \ \text{ for some } \ x_k \in B_{2nr_k}(x_{k-1})$$

where $r_k = C(\theta^{k-1}P)^{-\epsilon/n} = C\theta^{-(k-1)\epsilon/n}P^{-\epsilon/n}$.

To ensure $\{x_k\} \subset B_{1/2}$, we take $\sum 2nr_k < 1/4$. Hence, we choose M_0 so that

$$M_0^{\epsilon/n} \geq 8nC \sum_{k=1}^{\infty} \theta^{-(k-1)\epsilon/n} \ \text{ and } \ M_0 \geq \Big(\frac{C}{\theta - 1}\Big)^{n/(n+\epsilon)},$$

and choose $P > M_0$. This completes the proof. □

4.3 Schauder Estimates

In this section, we prove the Schauder estimates for viscosity solutions. Throughout this section, we always assume that $a^{ij}(x) \in C(B_1)$ satisfies

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for any $x \in B_1$ and any $\xi \in \mathbb{R}^n$.

We shall need the following approximation result. Namely, it states that if the coefficient matrix $(a^{ij}(x))$ is a "close" perturbation of the constant matrix $(a^{ij}(0))$ and thus is "close" to the identity matrix by the uniform ellipticity assumption, then the viscosity solution u is "close" to a solution of a Poisson equation at least locally.

Lemma 4.5. *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a^{ij}(x)D_{ij}u = f \ \text{ in } \ B_1$$

with $|u| \leq 1$ in B_1. Assume for some $\epsilon \in (0, 1/16)$,

$$\|a^{ij} - a^{ij}(0)\|_{L^n(B_{3/4})} \leq \epsilon.$$

Then there exists a function $h \in C(\bar{B}_{3/4})$ with $a^{ij}(0)D_{ij}h = 0$ in $B_{3/4}$ and $|h| \leq 1$ in $B_{3/4}$ for which

$$\|u - h\|_{L^\infty(B_{1/2})} \leq C(\epsilon^\gamma + \|f\|_{L^n(B_1)})$$

where $C > 0$ is a constant and $\gamma \in (0, 1)$ both depending only on $n, \lambda,$ and Λ.

Proof. We can certainly solve for such a harmonic function $h \in C(\bar{B}_{3/4}) \cap C^\infty(B_{3/4})$ where $a^{ij}(0)D_{ij}h = 0$ in $B_{3/4}$ and $h = u$ on $\partial B_{3/4}$. The maximum principle ensures $|h| \le 1$ in $B_{3/4}$ and note that u belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in B_1. Corollary 4.1 implies $u \in C^\alpha(\bar{B}_{3/4})$ for some $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$. Thus, from the global Schauder regularity theory for harmonic functions, the basic estimate

$$\|u\|_{C^\alpha(\bar{B}_{3/4})} \le C(n, \lambda, \Lambda)(1 + \|f\|_{L^n(B_1)})$$

implies

$$\|h\|_{C^{\alpha/2}(\bar{B}_{3/4})} \le C\|u\|_{C^\alpha(\bar{B}_{3/4})} \le C(n, \lambda, \Lambda)(1 + \|f\|_{L^n(B_1)}).$$

Since $u - h = 0$ on $\partial B_{3/4}$, we get for $\delta \in (0, 1/4)$,

$$\|u - h\|_{L^\infty(\partial B_{3/4-\delta})} \le C\delta^{\alpha/2}(1 + \|f\|_{L^n(B_1)}). \tag{4.13}$$

We claim that

$$\|D^2 h\|_{L^\infty(\partial B_{3/4-\delta})} \le C\delta^{\alpha/2-2}(1 + \|f\|_{L^n(B_1)}). \tag{4.14}$$

In fact, for any $x_0 \in B_{3/4-\delta}$, applying interior C^2 estimates on $h - h(x_1)$ in $B_\delta(x_0) \subset B_{3/4}$ for some $x_1 \in \partial B_\delta(x_0)$ yields

$$|D^2 h(x_0)| \le C\delta^{-2} \sup_{B_\delta(x_0)} |h - h(x_1)| \le C\delta^{-2}\delta^{\alpha/2}(1 + \|f\|_{L^n(B_1)}).$$

Note that $u - h$ is a viscosity solution of

$$a^{ij}(x)D_{ij}(u - h) = f(x) - (a^{ij}(x) - a^{ij}(0))D_{ij}h := F \text{ in } B_{3/4}.$$

So by the Alexandroff maximum principle and (4.13)-(4.14),

$$\begin{aligned}
\|u - h\|_{L^\infty(B_{3/4-\delta})} &\le \|u - h\|_{L^\infty(B_{3/4-\delta})} + C\|F\|_{L^n(B_{3/4-\delta})} \\
&\le \|u - h\|_{L^\infty(B_{3/4-\delta})} + C\|D^2\|_{L^\infty(B_{3/4-\delta})}\|a^{ij} - a^{ij}(0)\|_{L^n(B_{3/4})} + C\|f\|_{L^n(B_1)} \\
&\le C(\delta^{\alpha/2} + \delta^{\alpha/2-2}\epsilon)(1 + \|f\|_{L^n(B_1)}) + C\|f\|_{L^n(B_1)}.
\end{aligned}$$

The proof is complete once we take $\delta = \sqrt{\epsilon}$ and then $\gamma = \alpha/4$.

□

Definition 4.4. *A function g is Hölder continuous at 0 with exponent α in the L^n sense if*

$$[g]_{C^\alpha_{L^n}}(0) = \sup_{0 \le r \le 1} \frac{1}{r^\alpha}\Big(\frac{1}{|B_r|}\int_{B_r} |g(x) - g(0)|^n \, dx\Big)^{1/n} < \infty.$$

Theorem 4.3 (Schauder estimates)**.** *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a^{ij}(x)D_{ij}u = f \ \text{ in } \ B_1.$$

Assume (a^{ij}) is Hölder continous at 0 with exponent α in the L^n sense for some $\alpha \in (0,1)$. If f is Hölder continuous at 0 with exponent α in the L^n sense, then u is $C^{2,\alpha}$ at 0. Moreover, there exists a polynomial P of degree 2 such that

$$
\begin{aligned}
&|u-P|_{L^\infty(B_r(0))} \le C_* r^{2+\alpha} \text{ for any } r \in (0,1),\\
&|P(0)| + |DP(0)| + |D^2P(0)| \le C_*,\\
&C_* \le C(\|u\|_{L^\infty(B_1)} + |f(0)| + [f]_{C^\alpha_{L^n}}(0)),
\end{aligned}
$$

where $C>0$ is a constant depending only on $n, \lambda, \Lambda, \alpha$ and $[a^{ij}]_{C^\alpha_{L^n}}(0)$.

Proof. We organize the proof into two steps.
Step 1: Preparations We assume $f(0)=0$ otherwise we may consider $v = u - b^{ij}x_ix_jf(0)/2$ for some constant matrix (b^{ij}) such that $a^{ij}(0)b^{ij} = 1$. By scaling, we also assume that $[a^{ij}]_{C^\alpha_{L^n}}(0)$ is small. Next, by considering for $\delta > 0$,

$$\frac{u}{\|u\|_{L^\infty(B_1)} + \delta^{-1}[f]_{C^\alpha_{L^n}}(0)},$$

we may also assume $\|u\|_{L^\infty(B_1)} \le 1$ and $[f]_{C^\alpha_{L^n}}(0) \le \delta$.
Step 2: Suppose $u \in C(B_1)$ is a viscosity solution of

$$a^{ij}(x)D_{ij}u = f \text{ in } B_1$$

with

$$\|u\|_{L^\infty(B_1)} \le 1,\ [a^{ij}]_{C^\alpha_{L^n}}(0) \le \delta$$

and

$$\Big(\frac{1}{|B_r|}\int_{B_r} |f|^n\,dx\Big)^{1/n} \le \delta r^\alpha \text{ for any } r \in (0,1).$$

We claim there exists a constant $\delta > 0$, depending only on $n, \lambda, \Lambda,$ and α and a polynomial P of degree 2 with

$$\|u-P\|_{L^\infty(B_r)} \le Cr^{2+\alpha} \text{ for any } r \in (0,1), \tag{4.15}$$

and

$$|P(0)| + |DP(0)| + |D^2P(0)| \le C(n,\lambda,\Lambda,\alpha). \tag{4.16}$$

First, we show there exist $\mu \in (0,1)$, depending only on $n, \lambda, \Lambda,$ and α, and a sequence of polynomials of degree 2,

$$P_k(x) = a_k + b_k \cdot x + (1/2)x^TC_kx,$$

such that for any $k = 0,1,2,\ldots,$

$$a^{ij}(0)D_{ij}P_k = 0,\ \|u-P_k\|_{L^\infty(B_{\mu^k})} \le \mu^{k(2+\alpha)}, \tag{4.17}$$

and

$$|a_k - a_{k-1}| + \mu^{k-1}|b_k - b_{k-1}| + \mu^{2(k-1)}|C_k - C_{k-1}| \le C\mu^{(k-1)(2+\alpha)}. \tag{4.18}$$

Note that $P_0, P_{-1} \equiv 0$ and C is a constant depending only on n, λ, Λ, and α.

Obviously, the theorem follows from (4.17)-(4.18) since a_k, b_k and C_k converge to some a, b and C, and the limiting polynomial,

$$P(x) = a + b \cdot x + (1/2)x^T Cx,$$

satisfies

$$|P_k(x) - P(x)| \le C(|x|^2\mu^{\alpha k} + |x|\mu^{(\alpha+1)k} + \mu^{(\alpha+2)k}) \le C\mu^{(2+\alpha)k}$$

for any $|x| \le \mu^k$. Hence, for $|x| \le \mu^k$,

$$|u(x) - P(x)| \le |u(x) - P_k(x)| + |P_k(x) - P(x)| \le C\mu^{(2+\alpha)k},$$

which implies

$$|u(x) - P(x)| \le C|x|^{2+\alpha} \text{ for any } x \in B_1.$$

Therefore, it only remains to prove (4.17) and (4.18), and we do so by induction. The initial step $k = 0$ is clearly true. Assume both estimates hold for $k = 0, 1, \ldots, \ell$. We prove the next step $k = \ell + 1$ holds. Consider the function

$$\tilde{u}(y) = \frac{1}{\mu^{\ell(2+\alpha)}}(u - P_\ell)(\mu^\ell y) \text{ for } y \in B_1.$$

Then $\tilde{u}$ belongs to $C(B_1)$ and is a viscosity solution of

$$\tilde{a}^{ij}(x)D_{ij}\tilde{u} = \tilde{f} \text{ in } B_1$$

where

$$\tilde{a}^{ij}(y) = \mu^{-\ell\alpha}a^{ij}(\mu^\ell y),$$

and

$$\tilde{f}(y) = \mu^{-\ell\alpha}(f(\mu^\ell y) - a^{ij}(\mu^\ell y)D_{ij}P_k).$$

We want to apply Lemma 4.16. So we check that

$$\|\tilde{a}^{ij} - \tilde{a}^{ij}(0)\|_{L^n(B_1)} \le \mu^{-\ell\alpha}\|a^{ij} - a^{ij}(0)\|_{L^n(B_{\mu^\ell})} \le [a^{ij}]^\alpha_{L^n}(0) \le \delta,$$

and

$$\|\tilde{f}\|_{L^n(B_1)} \le \mu^{-\ell\alpha}\|f\|_{L^n(B_{\mu^\ell})} + \mu^{-\ell\alpha}\sup|D^2P_\ell|\|a^{ij} - a^{ij}(0)\|_{L^n(B_{\mu^\ell})} \le \delta + C\delta$$

where we used

$$|D^2P_\ell| \le \sum_{k=1}^{\ell}|D^2P_k - D^2P_{k-1}| \le \sum_{k=1}^{\ell}\mu^{(k-1)\alpha} \le C.$$

Taking $\epsilon = C(n, \lambda, \Lambda)\delta$ in Lemma 4.16, we can find $h \in C(\bar{B}_{3/4})$ with $\tilde{a}^{ij}(0)D_{ij}h = 0$ in $B_{3/4}$ and $|h| \le 1$ in $B_{3/4}$ such that

$$\|\tilde{u} - h\|_{L^\infty(B_{1/2})} \le C(\epsilon^\gamma + \epsilon) \le 2C\epsilon^\gamma.$$

Write $\tilde{P}(y) = h(0) + Dh(0) + y^T D^2 h(0) y/2$. Then the interior estimates for h yield

$$\|\tilde{u} - \tilde{P}\|_{L^\infty(B_\mu)} \le \|\tilde{u} - h\|_{L^\infty(B_\mu)} + \|h - \tilde{P}\|_{L^\infty(B_\mu)} \le 2C\epsilon^\gamma + C\mu^3 \le \mu^{2+\alpha}$$

by choosing μ small and then ϵ small accordingly. Rescaling back, we get

$$|u(x) - P_\ell(x) - \mu^{\ell(2+\alpha)}\tilde{P}(\mu^{-\ell}x)| \le \mu^{(\ell+1)(2+\alpha)} \text{ for any } x \in B_{\mu^{\ell+1}}.$$

This implies (4.17) for $k = \ell + 1$ if we take

$$P_{k+1}(x) = P_k(x) + \mu^{\ell(2+\alpha)}\tilde{P}(\mu^{-\ell}x).$$

Estimate (4.18) follows easily. □

We also have the following Cordes-Nirenberg type estimate, but we omit its proof.

Theorem 4.4 (Cordes-Nirenberg)**.** *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a^{ij}(x)D_{ij}u = f \text{ in } B_1.$$

Then for any $\alpha \in (0,1)$, there exists an $\theta > 0$ depending only on n, λ, Λ, and α such that if

$$\Big(\frac{1}{|B_r|}\int_{B_r} |a^{ij}(x) - a^{ij}(0)|^n \, dx\Big)^{1/n} \le \theta \text{ for any } f \in (0,1),$$

then u is $C^{1,\alpha}$ at 0. Namely, there exists an affine function L such that

$$\begin{aligned}
&|u - L|_{L^\infty(B_r(0))} \le C_* r^{1+\alpha} \text{ for any } r \in (0,1),\\
&|L(0)| + |DL(0)| \le C_*,\\
&C_* \le C\Big\{\|u\|_{L^\infty(B_1)} + \sup_{0<r<1}\Big(\frac{1}{|B_r|}\int_{B_r} |f(x)|^n \, dx\Big)^{1/n}\Big\},
\end{aligned}$$

where $C > 0$ is a constant depending only on n, λ, Λ, and α.

4.4 $W^{2,p}$ Estimates

In this section, we assume throughout that $f \in C(B_1)$, $(a^{ij}) \in C(B_1)$ and there exist $\lambda, \Lambda > 0$ such that

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$$

for any $x \in U$ and any $\xi \in \mathbb{R}^n$. Our main result here is the following

Theorem 4.5. *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a^{ij}(x)D_{ij}u = f \ \ in \ \ B_1.$$

For any $p \in (n, \infty)$, there exists an $\epsilon > 0$ depending only on n, λ, Λ, and p such that if

$$\Big(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |a^{ij}(x) - a^{ij}(x_0)|^n \, dx\Big)^{1/n} \leq \epsilon \ \ for\ any\ B_r(x_0) \subset B_1,$$

then $u \in W^{2,p}_{loc}(B_1)$. Moreover,

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}),$$

where $C > 0$ is a constant depending only on n, λ, Λ, and p.

As before, it suffices to prove the following.

Theorem 4.6. *Suppose $u \in C(B_{8\sqrt{n}})$ is a viscosity solution of*

$$a^{ij}(x)D_{ij}u = f \ \ in \ \ B_{8\sqrt{n}}.$$

For any $p \in (n, \infty)$, there exist $\epsilon > 0$ and $C > 0$ depending only on n, λ, Λ, and p such that if

$$\|u\|_{L^\infty(B_{8\sqrt{n}})} \leq 1 \ \ and \ \ \|f\|_{L^p(B_{8\sqrt{n}})} \leq \epsilon$$

and if

$$\Big(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |a^{ij}(x) - a^{ij}(x_0)|^n \, dx\Big)^{1/n} \leq \epsilon \ \ for\ any\ B_r(x_0) \subset B_{8\sqrt{n}},$$

then $u \in W^{2,p}(B_1)$ and $\|u\|_{W^{2,p}(B_1)} \leq C$.

CHAPTER 5

The Method of Moving Planes and Its Variants

In this chapter, we introduce a powerful tool used to study the properties of solutions for semilinear elliptic equations. The method is called the method of moving planes and it originated from Alexandroff in his study of embedded constant mean curvature surfaces. It was further developed in the works of Serrin [30] and Gidas, Ni and Nirenberg [12] and later adapted to many other problems involving differential and integral equations (see [6] and the references therein). We will focus on applying this method to obtain symmetry and monotonicity results for positive solutions of the Lane-Emden equation and we shall essentially adopt the framework of Chen and Li [5].

Consider the following semilinear elliptic problem

$$-\Delta u = u^p, \;\; x \in \mathbb{R}^n, \;\;\; n \geq 3. \tag{5.1}$$

Our goal is to prove the following main result.

Theorem 5.1. *For $p = (n+2)/(n-2)$, every positive C^2 solution of equation (5.1) must be radially symmetric and monotone decreasing about some point, and thus assumes the form*

$$u(x) = \frac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2 + |x - x^0|^2)^{\frac{n-2}{2}}} \;\; \textit{for some} \;\; \lambda > 0 \;\; \textit{and} \;\; x^0 \in \mathbb{R}^n.$$

For $1 < p < (n+2)/(n-2)$, the only non-negative C^2 solution of equation (5.1) is the trivial one, $u \equiv 0$.

5.1 Preliminaries

We first start by introducing some necessary tools for the method of moving planes. Namely, we introduce the Kelvin transform and various comparison theorems, i.e., maximum principles, for elliptic problems on unbounded domains. First, the Kelvin transform of the function u, which we denote by $\bar{u}$, is given by

$$\bar{u}(x) = \frac{1}{|x|^{n-2}} u\Big(\frac{x}{|x|^2}\Big).$$

Then, if u is a solution of equation (5.1), then $\bar{u}$ is a solution of

$$-\Delta \bar{u} = |x|^{p(n-2)-(n+2)} \bar{u}^p, \quad x \in \mathbb{R}^n \backslash \{0\}. \tag{5.2}$$

Now we revisit some variants of the maximum and comparison principle, which are essential ingredients in the method of moving planes. These results have been covered in Chapter 1, but we state them again here for convenience. The first theorem is an extension of Hopf's Lemma and the strong maximum principle for domains that are not necessarily bounded and for lower order terms that need not be positive (note this version holds for the Dirichlet problem involving a perturbed operator of the Laplacian). The second theorem is the maximum principle based on comparisons, which was stated and proved earlier in Section 1.2.2 of Chapter 1. We state it here for convenience.

Theorem 5.2 (Maximum principle and Hopf's lemma for possibly unbounded domains)**.** *Let U be a domain in $\mathbb{R}^n$ with smooth boundary ∂U, and assume $u \in C^2(U) \cap C(\bar{U})$ satisfies*

$$\begin{cases} -\Delta u + \sum_{i=1}^n b^i(x) D_i u + c(x) u \geq 0 & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{5.3}$$

where $b^i(x)$ and $c(x)$ are bounded functions. Then the following hold.

(a) If u vanishes at some point in U, then $u \equiv 0$ in U;

(b) If u is non-trivial in U, then $\partial u / \partial \nu < 0$ on ∂U.

The next result is a useful comparison principle that applies to possibly unbounded domains.

Theorem 5.3 (Maximum principle based on comparisons)**.** *Assume that U is a bounded domain. Let ϕ be a positive function on $\bar{U}$ satisfying*

$$-\Delta \phi + \lambda(x) \phi \geq 0. \tag{5.4}$$

Assume that u is a classical solution of

$$\begin{cases} -\Delta u + c(x) u \geq 0 & \text{in } U, \\ u \geq 0 & \text{on } \partial U. \end{cases} \tag{5.5}$$

If

$$c(x) > \lambda(x) \ \text{ for all } x \in U, \tag{5.6}$$

then

$$u \geq 0 \;\; in \;\; U.$$

If U is unbounded, then the result remains true provided that the following additional condition is assumed:

$$\liminf_{|x|\longrightarrow\infty} \frac{u(x)}{\phi(x)} \geq 0. \tag{5.7}$$

In our application of the above theorem, we will consider two cases:

(a) U is a "narrow" region,

(b) the coefficient $c(x)$ has sufficient decay at infinity.

First, we examine when U is a narrow region; namely, let us consider the narrow strip with width $\ell > 0$, i.e.,

$$U = \{x \in \mathbb{R}^n \,|\, 0 < x_1 < \ell\}.$$

We can take $\varphi(x) = \sin((x_1 + \epsilon)/\ell)$ so that $-\Delta\varphi = (1/\ell)^2\varphi$. Thus, $\lambda(x) = -(1/\ell)^2$, which can be "very negative" if ℓ is suitably small.

Corollary 5.1 (Narrow region)**.** *If u satisfies* (5.5) *with bounded function $c(x)$, the width ℓ of the region U is sufficiently small, $c(x)$ satisfies* (5.6)*, i.e., $c(x) > \lambda(x) = -1/\ell^2$, then*

$$u \geq 0 \;\; in \;\; U.$$

In the case of (b) with $n \geq 3$, we can choose a positive number $q < n - 2$ and take $\phi(x) = |x|^{-q}$, then a simple calculation yields

$$-\Delta\phi = \frac{q(n-2-q)}{|x|^2}\phi := -\lambda(x)\phi.$$

Therefore, if $c(x)$ has sufficient decay, the previous theorem implies the following.

Corollary 5.2 (Decay at infinity)**.** *Assume there exists $R > 0$ such that*

$$c(x) > -\frac{q(n-2-q)}{|x|^2}, \;\; for\ all \;\; |x| > R. \tag{5.8}$$

Suppose that

$$\lim_{|x|\to\infty} u(x)|x|^q = 0.$$

Let U be a region contained in $B_R^C(0)$. If u satisfies (5.5) *on $\bar{U}$, then*

$$u(x) \geq 0 \;\; for\ all \;\; x \in U.$$

5.2 The Proof of Theorem 5.1

We are now ready to prove Theorem 5.1.

Proof. Set $p = (n+2)/(n-2)$ and we shall first impose a fast decay assumption on the solution, i.e.,

$$u(x) = O(|x|^{-(n-2)}). \tag{5.9}$$

Define

$$\Sigma_\lambda := \Big\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \,|\, x_1 < \lambda\Big\} \text{ and } T_\lambda := \partial\Sigma_\lambda$$

and let x^λ be the reflection point of x about the plane T_λ, i.e.,

$$x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n).$$

Define

$$u_\lambda(x) := u(x^\lambda) \text{ and } w_\lambda(x) := u_\lambda(x) - u(x).$$

Step 1: Prepare to move the plane near $-\infty$.

Namely, we will show that we can find $N > 0$ suitably large so that if $\lambda \leq -N$,

$$w_\lambda(x) \geq 0 \text{ for all } x \in \Sigma_\lambda. \tag{5.10}$$

Indeed, the mean value theorem implies

$$-\Delta w_\lambda(x) = u_\lambda^p(x) - u^p(x) = p\psi_\lambda^{p-1} w_\lambda(x), \tag{5.11}$$

where $\psi_\lambda(x)$ is some number between $u_\lambda(x)$ and $u(x)$. In view of Theorem 5.3 and Corollary 5.2, we take $c(x) = -p\psi_\lambda^{p-1}(x)$ and see that (5.10) holds provided we show $c(x)$ has sufficient decay at infinity at the points $\tilde{x}$ where $w_\lambda(\tilde{x}) < 0$. Well, at these points, we have

$$u_\lambda(\tilde{x}) < u(\tilde{x})$$

and so

$$0 \leq u_\lambda(\tilde{x}) \leq \psi_\lambda(\tilde{x}) \leq u(\tilde{x}).$$

Indeed, by assumption (5.9) and since $p = (n+2)/(n-2)$,

$$\psi_\lambda^{p-1}(\tilde{x}) = O\Big((|\tilde{x}|^{-(n-2)})^{\frac{4}{n-2}}\Big) = O(|\tilde{x}|^{-4})$$

and the decay of the coefficient is greater than 2 as required in Corollary 5.2, which implies the desired result. Namely, we can find $N > 0$ sufficiently large so that for $\lambda \leq -N$ (or $|\tilde{x}|$ sufficiently large), we must have (5.10).

Step 2: Moving the Plane.

We can increase the value of λ, and thus move the plane T_λ to the right, provided inequality (5.10) holds. Define

$$\lambda_0 := \sup\{\lambda \,|\, w_\lambda(x) \geq 0, \;\text{ for all }\; x \in \Sigma_\lambda\}.$$

Clearly, $\lambda_0 < \infty$ due to the asymptotic behavior of u for x_1 near ∞. We claim that

$$w_{\lambda_0} \equiv 0 \;\text{ in }\; \Sigma_{\lambda_0}. \tag{5.12}$$

Otherwise, the strong maximum principle on unbounded domains would imply that

$$w_{\lambda_0}(x) > 0 \;\text{ for all }\; x \in interior(\Sigma_{\lambda_0}). \tag{5.13}$$

Assuming that this holds true, we claim that we can then move the plane T_{λ_0} further to the right a small distance, thereby contradicting the definition of λ_0 and conclude that (5.12) holds. Namely, we claim there exists a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, we have that

$$w_{\lambda_0+\delta}(x) \geq 0 \;\text{ for all }\; x \in \Sigma_{\lambda_0+\delta}. \tag{5.14}$$

At first glance, one may assume that this would follow from Corollary 5.1, however, we cannot apply this directly since we are not able to guarantee that w_{λ_0} is bounded away from 0 on the left boundary of the narrow region. To circumvent this, we apply Corollary 5.2 instead but to a carefully chosen auxiliary function. Namely, we set

$$\bar{w}_\lambda(x) = \frac{w_\lambda(x)}{\phi(x)},$$

where

$$\phi(x) = |x|^{-q} \;\text{ with }\; q \in (0, n-2).$$

Then, a direct calculation will show that

$$-\Delta \bar{w}_\lambda = 2D\bar{w}_\lambda \cdot \frac{D\phi}{\phi} + \Big(-\Delta w_\lambda + \frac{\Delta \phi}{\phi} w_\lambda \Big) \frac{1}{\phi}. \tag{5.15}$$

Claim: There exists $R_0 > 0$, independent of λ, such that if x^0 is a minimum point of $\bar{w}_\lambda$ and $\bar{w}_\lambda(x^0) < 0$, then $|x^0| < R_0$.

To show this claim holds, we proceed by contradiction. Assume that x^0 is a negative minimum of $\bar{w}_\lambda$ but that $|x^0|$ can be chosen to be suitably large. Thus,

$$-\Delta \bar{w}_\lambda(x^0) \leq 0, \tag{5.16}$$

and

$$D\bar{w}_\lambda(x^0) = 0. \tag{5.17}$$

By the asymptotic behavior of u at infinity and since $|x^0|$ is sufficiently large,

$$c(x^0) := -p\psi_\lambda(x^0)^{p-1} > -\frac{q(n-2-q)}{|x^0|^2} \equiv \frac{\Delta\phi(x^0)}{\phi(x^0)}.$$

It follows from (5.11) and $w_\lambda(x_0) < 0$ that

$$0 = -\Delta w_\lambda(x^0) + c(x^0)w_\lambda(x^0) < -\Delta w_\lambda(x^0) + \frac{\Delta\phi(x^0)}{\phi(x^0)}w_\lambda(x^0).$$

Hence,

$$\Big(-\Delta w_\lambda + \frac{\Delta\phi}{\phi}w_\lambda\Big)(x^0) > 0.$$

Combining this with (5.15) and (5.17) leads to $-\Delta\bar{w}_\lambda(x^0) > 0$, which contradicts with (5.16). This completes the proof of the claim.

Hence, if (5.14) is violated for any $\delta > 0$, then we can find a sequence of positive numbers $\{\delta_i\} \longrightarrow 0$ where for each i, we denote the corresponding negative minimum of $\bar{w}_{\lambda_0+\delta_i}$ by x^i. Then, by the last claim, we have $|x^i| \leq R_0$ for $i = 1, 2, 3, \ldots$. Then, by compactness, we can extract a subsequence, which we still denote by $\{x^i\}$, that converges to some point $x^0 \in \mathbb{R}^n$. Hence,

$$D\bar{w}_{\lambda_0}(x^0) = \lim_{i\to\infty} D\bar{w}_{\lambda_0+\delta_i}(x^i) = 0,$$
$$\bar{w}_{\lambda_0}(x^0) = \lim_{i\to\infty} \bar{w}_{\lambda_0+\delta_i}(x^i) \leq 0.$$

From this, we deduce that $\bar{w}_{\lambda_0}(x^0) = 0$, since we also know that $\bar{w}_{\lambda_0} \geq 0$. Moreover,

$$Dw_{\lambda_0}(x^0) = D\bar{w}_{\lambda_0}(x^0)\phi(x^0) + \bar{w}_{\lambda_0}(x^0)D\phi(x^0) = 0. \tag{5.18}$$

In view of (5.13) and the fact that $w_{\lambda_0}(x^0) = 0$, we must have that x^0 lies on the boundary of Σ_{λ_0}. Then Hopf's lemma of Theorem 5.2 indicates that

$$\frac{\partial w_{\lambda_0}}{\partial\nu}(x^0) < 0,$$

which contradicts with (5.18) and we conclude that $w_{\lambda_0} \equiv 0$ or that $u(x) = u_{\lambda_0}(x)$ for all $x \in \Sigma_{\lambda_0}$.

So far, we have shown that u is symmetric and monotone decreasing about the plane T_{λ_0}. Since the coordinate axis x_1 can be chosen arbitrarily, we conclude that u must be radially symmetric and monotone decreasing about some point. Moreover, basic uniqueness theory for ordinary differential equations imply that u must have the form as described in the theorem.

Step 3: Removing the fast decay assumption.

Apply the Kelvin transform on the solution $u(x)$ to get $v(x)$:

$$v(x) = \frac{1}{|x|^{n-2}}u\Big(\frac{x}{|x|^2}\Big).$$

Then v has the fast decay at infinity and satisfies the following semilinear equation in punctured space,

$$-\Delta v = v^p \text{ in } \mathbb{R}^n \backslash \{0\}.$$

We can apply the same arguments of Steps 1 and 2, but after some careful modifications. More precisely, we must carry out the moving plane procedure on $\Sigma_\lambda \backslash \{x^\lambda\}$ to avoid the possible singularity at the origin introduced by the Kelvin transform. Equivalently, the function $w_\lambda(x) = v_\lambda(x) - v(x)$ has a possible singularity at $(2\lambda, 0, \dots, 0)$. We just have to show that all the points of interest that arise in applying the earlier arguments actually occur away from the possible singularity and we can carry out the method as usual.

Just as before, we want to show that $w_{\lambda_0} \equiv 0$ for $x \in \Sigma_{\lambda_0} \backslash \{x^{\lambda_0}\}$. Assume the contrary. In fact, without loss of generality, we can assume that $\lambda_0 \leq 0$ and $w_{\lambda_0} \not\equiv 0$ in $\Sigma_{\lambda_0} \backslash \{x^{\lambda_0}\}$. Thus, the maximum principle (see Theorem 5.2) implies that $w_{\lambda_0}(x) > 0$ for $x \in \Sigma_{\lambda_0} \backslash \{x^{\lambda_0}\}$. Now suppose $\delta_i \longrightarrow 0$ is a sequence of positive reals such that $\bar{w}_{\lambda_0+\delta_i}(x) < 0$ for some $x \in \Sigma_{\lambda_0} \backslash \{x^{\lambda_0}\}$. We need to show that for each i, the negative infimum of $\bar{w}_{\lambda_0+\delta_i}(x)$ is achieved at some point $x^i \in \Sigma_{\lambda_0} \backslash \{x^{\lambda_0}\}$, and that the sequence of points $\{x^i\}$ is bounded away from the singularities $x^{\lambda_0+\delta_i}$ of $w_{\lambda_0+\delta_i}$. Indeed, this is guaranteed by the following two facts.

(a) There exist $\epsilon > 0$ and $\delta > 0$ such that

$$\bar{w}_{\lambda_0}(x) \geq \epsilon \text{ for } x \in B_\delta(x^{\lambda_0}) \backslash \{x^{\lambda_0}\}.$$

(b) There holds

$$\lim_{\lambda \to \lambda_0} \inf_{x \in B_\delta(x^\lambda)} \bar{w}_\lambda(x) \geq \inf_{x \in B_\delta(x^{\lambda_0})} \bar{w}_{\lambda_0}(x) \geq \epsilon.$$

For (a), since $\bar{w}_{\lambda_0} > 0$ in $\Sigma_{\lambda_0} \backslash \{x^{\lambda_0}\}$ and $\Delta w_{\lambda_0} < 0$, we may compare $\bar{w}_{\lambda_0}$ with a harmonic function h satisfying

$$\begin{cases} \Delta h = 0 & \text{in } B_\delta(x^{\lambda_0}) \backslash \{x^{\lambda_0}\}, \\ h = \epsilon & \text{on } \partial B_\delta(x^{\lambda_0}) \backslash \{x^{\lambda_0}\}, \end{cases}$$

for suitably small $\epsilon > 0$ and $\delta > 0$ such that $\bar{w}_{\lambda_0} \geq \epsilon$ on $\partial B_\delta(x^{\lambda_0}) \backslash \{x^{\lambda_0}\}$. Hence, $z(x) := \bar{w}_{\lambda_0}(x) - h(x)$ satisfies

$$\begin{cases} -\Delta z \geq 0 & \text{in } B_\delta(x^{\lambda_0}) \backslash \{x^{\lambda_0}\}, \\ z \geq 0 & \text{on } \partial B_\delta(x^{\lambda_0}) \backslash \{x^{\lambda_0}\}, \end{cases}$$

and the maximum principle implies $z \geq 0$ or $\bar{w}_{\lambda_0} \geq \epsilon$ in $B_\delta(x^{\lambda_0}) \backslash \{x^{\lambda_0}\}$. So any negative infimum must be attained away from the singularity. Part (b) follows easily from part (a). Then, following the same ideas found in the first two steps, we can derive a contradiction. Hence, we then deduce that $w_{\lambda_0} \equiv 0$ on $\Sigma_{\lambda_0} \backslash \{x^{\lambda_0}\}$ and so v is radially symmetric and monotone decreasing about some point x^0 in $\mathbb{R}^n$. If x^0 is not the origin, then the origin is a regular point and u has the fast decay property at infinity to begin with and we are done. Otherwise, if $x_0 = 0$ and thus v is symmetric and monotone about the origin, then

u is also symmetric and monotone about the origin since it is easy to check that $u(x) = |x|^{-(n-2)}v(x/|x|^2)$. This completes the proof of the classification result.

Step 4: Liouville property in the subcritical case.

It remains to prove that $u \equiv 0$ in the subcritical case $p \in (1, \frac{n+2}{n-2})$. Again, by the Kelvin transform, we have that v, as defined earlier, is now a solution of

$$-\Delta v = |x|^{p(n-2)-(n+2)}v^p \text{ in } \mathbb{R}^n \backslash \{0\}. \tag{5.19}$$

Since the subcritical condition implies that $p(n-2)-(n+2) < 0$, the coefficient of equation (5.19) decays at infinity. Therefore, we may apply the method of moving planes, i.e., Steps 1–3, to get that v is radially symmetric and monotone decreasing about some point $x^0 \in \mathbb{R}^n$. In fact, it is clear that $x^0 = 0$ due to the singular coefficient of equation (5.19). Thus, it is easy to see that u is also radially symmetric and monotone decreasing about the origin. Then, as a consequence of the well-known Pohozaev type identity for equation (5.19), $u \equiv 0$. Alternatively, we can argue, using the translation and dilation invariance of equation (5.19), that v must actually be constant and therefore trivial. This completes the proof of the theorem. □

Remark 5.1. *In the supercritical case* $p > (n+2)/(n-2)$, *the coefficient in* (5.19) *no longer decays since the* $p(n-2)-(n+2) > 0$. *This destroys the mechanism for carrying out the method of moving planes, since we are not able to get the correct inequality in* (5.11). *The classification of positive solutions in the supercritical case remains open.*

Remark 5.2. *We see that the "decay at infinity" principle is important in applying the method of moving planes to the Lane-Emden equation in* $\mathbb{R}^n$, *but we did not make use of the "narrow region" principle. Indeed, the narrow region principle is more appropriate for certain bounded domains. Namely, it is a key ingredient in applying the method of moving planes for radially symmetric, bounded domains. A consequence of this is the following result whose proof we omit.*

Theorem 5.4. *Assume that* f *is a Lipschitz continuous function such that*

$$|f(p) - f(q)| \leq C_0|p-q|$$

for some positive constant C_0. *Then every positive solution* $u \in C^2(B_1(0)) \cap C(\bar{B}_1(0))$ *of*

$$\begin{cases} -\Delta u = f(u) & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0), \end{cases}$$

is radially symmetric and monotone decreasing about the origin.

5.3 The Method of Moving Spheres

In this section, we introduce a variant of the method of moving planes known as the method of moving spheres. This alternative technique uses the inversion of the Kelvin transform on spheres and invokes comparison theorems to obtain symmetry and monotonicity properties of solutions to certain elliptic problems. The advantage of this approach is that we can deduce the classification and Liouville theorems for non-negative solutions in one fell swoop. This is, in some sense, more direct than the method of moving planes, which first establishes the radial symmetry and monotonicity properties then reduces the problem into an ODE one to arrive at the desired results. The moving sphere approach is also advantageous in certain domains such as half-spaces.

First, we state and prove two fundamental calculus lemmas that are important ingredients in the method of moving spheres.

Lemma 5.1. *Let $f \in C^1(\mathbb{R}^n)$, $n \geq 1$ and $\nu > 0$. Suppose that for each $x \in \mathbb{R}^n$, there exists $\lambda = \lambda(x)$ such that*

$$\Big(\frac{\lambda(x)}{|y-x|}\Big)^{\nu} f\Big(x + \lambda(x)^2 \frac{y-x}{|y-x|^2}\Big) = f(y), \ \ y \in \mathbb{R}^n \backslash \{x\}. \tag{5.20}$$

Then for some $a \geq 0$, $d > 0$, and $x_0 \in \mathbb{R}^n$,

$$f(x) = \pm \Big(\frac{a}{d + |x - x_0|^2}\Big)^{\nu/2}.$$

Proof. From (5.20), we have that

$$\ell := \lim_{|y| \to \infty} |y|^{\nu} f(y) = \lambda(x)^{\nu} f(x), \ \ x \in \mathbb{R}^n.$$

If $\ell = 0$, then $f \equiv 0$ and we are done. However, if $\ell \neq 0$, then f does not change sign. Therefore, without loss of generality, we may take $\ell = 1$ and f positive. For large y, taking Taylor expansions of the left-hand side of (5.20) at 0 and x yield

$$f(y) = \Big(\frac{\lambda(0)}{|y|}\Big)^{\nu} \Big(f(0) + \frac{\partial f}{\partial y_i}(0)\lambda(0)^2 \frac{y_i}{|y|^2} + o(|y|^{-1})\Big) \tag{5.21}$$

and

$$f(y) = \Big(\frac{\lambda(x)}{|y-x|}\Big)^{\nu} \Big(f(x) + \frac{\partial f}{\partial y_i}(x)\lambda(x)^2 \frac{y_i - x_i}{|y-x|^2} + o(|y|^{-1})\Big), \tag{5.22}$$

where $o(|y|^{-1})$ represents some higher-order term such that $o(|y|^{-1})/|y|^{-1} \longrightarrow 0$ as $|y| \longrightarrow \infty$. From our assumption that $\ell = 1$, we combine (5.20), which implies $\lambda(x) = f(x)^{-1/\nu}$, with (5.21) and (5.22) to get

$$f(x)^{-1-2/\nu} \frac{\partial f}{\partial y_i}(x) = f(0)^{-1-2/\nu} \frac{\partial f}{\partial y_i}(0) - \nu x_i.$$

Integrating this yields that for some $x_0 \in \mathbb{R}^N$, $d > 0$,

$$f(x)^{-2/\nu} = |x - x_0|^2 + d.$$

Solving for $f(x)$ will finish the proof. □

Lemma 5.2. *Let $f \in C^1(\mathbb{R}^n)$, $n \geq 1$, and $\nu > 0$. Suppose that*

$$\Big(\frac{\lambda}{|y-x|}\Big)^{\nu} f\Big(x + \lambda \frac{y-x}{|y-x|^2}\Big) \leq f(y), \ \textit{for all } \lambda > 0, \, x \in \mathbb{R}^n, \, |y-x| \geq \lambda. \tag{5.23}$$

Then $f \equiv$ constant.

Proof. For $x \in \mathbb{R}^n$, $\lambda > 0$, define

$$g_{x,\lambda}(z) = f(x+z) - \Big(\frac{\lambda}{|z|}\Big)^{\nu} f\Big(x + \lambda^2 \frac{z}{|z|^2}\Big), \ |z| \geq \lambda.$$

Indeed, $g_{x,|z|}(z) = 0$ and $g_{x,|z|}(rz) \geq 0$ for $r \geq 1$. Then, it follows that

$$\frac{d}{dr} g_{x,|z|}(rz)\Big|_{r=1} \geq 0.$$

Hence, a direct calculation yields

$$2Df(z+x) \cdot z + \nu f(z+x) \geq 0.$$

Since z and x are chosen arbitrarily, a change of variables shows that

$$2Df(y) \cdot (y-x) + \nu f(y) \geq 0.$$

Multiplying the preceding inequality by $|x|^{-1}$ and sending $|x| \longrightarrow \infty$, we conclude that $Df(y) \cdot \theta \leq 0$ for all $y \in \mathbb{R}^n$ and $\theta \in \mathbb{S}^{n-1}$. Hence, $Df \equiv 0$ in $\mathbb{R}^n$, and this completes the proof. □

We give an alternative proof of Theorem 5.1 using the method of moving spheres. We interrupt momentarily for some notation. For $x \in \mathbb{R}^n$ and $\lambda > 0$, define the Kelvin transformation of u by

$$u_{x,\lambda}(y) = \Big(\frac{\lambda}{|y-x|}\Big)^{n-2} u\Big(x + \lambda^2 \frac{y-x}{|y-x|^2}\Big), \ y \in \mathbb{R}^n \backslash \{x\}.$$

The following lemma ensures that we may start the moving sphere procedure.

Lemma 5.3. *For every $x \in \mathbb{R}^n$, there exists $\lambda(x) > 0$ such that $u_{x,\lambda(x)}(y) \leq u(y)$.*

From this we may define the following value $\lambda_0 \in (0, \infty]$. For each $x \in \mathbb{R}^n$ we set

$$\lambda_0(x) = \sup\{\mu > 0 \,|\, u_{x,\lambda}(y) \leq u(y), \ \text{for all } |y-x| \geq \lambda, \, \lambda \in (0, \mu]\}.$$

Proof of Theorem 5.1. We consider the two cases separately.
Critical case: Let $p = (n+2)/(n-2)$ and suppose that u is a positive solution of (5.1).

Step 1: We claim that if $\lambda_0(x) < \infty$ for some point $x \in \mathbb{R}^n$, then

$$u_{x,\lambda_0(x)} \equiv u \text{ in } \mathbb{R}^n \backslash \{0\}.$$

Without loss of generality, we may take $x = 0$ and $\lambda_0 = \lambda_0(0)$, $u_\lambda = u_{0,\lambda}$, and

$$\Sigma_\lambda = \{y \in \mathbb{R}^n \mid |y| > \lambda\}.$$

From the definition of λ_0,

$$u \geq u_{\lambda_0} \text{ on } \Sigma_{\lambda_0}.$$

Recall that the Kelvin transform of u satisfies

$$-\Delta u_\lambda = u_\lambda^{\frac{n+2}{n-2}}, \ \lambda > 0.$$

So by setting $w_\lambda = u - u_\lambda$, we get

$$-\Delta w_{\lambda_0} = u^{\frac{n+2}{n-2}} - u_{\lambda_0}^{\frac{n+2}{n-2}} \geq 0 \text{ in } \Sigma_{\lambda_0}.$$

If $w_{\lambda_0} \equiv 0$ in Σ_{λ_0}, then we are done. Otherwise, Hopf's lemma and the compactness of $\partial B_{\lambda_0}(0)$ imply that

$$\frac{d}{dr} w_{\lambda_0}\Big|_{\partial B_{\lambda_0}(0)} \geq c > 0.$$

By the continuity of Du, there exists $R \geq \lambda_0$ such that

$$\frac{d}{dr} w_\lambda \geq c/2 > 0, \text{ for } \lambda \in [\lambda_0, R],\ r \in [\lambda, R].$$

Thus, since $w_\lambda \equiv 0$ on $\partial B_\lambda(0)$, we have

$$w_\lambda(y) > 0 \text{ for } \lambda \in [\lambda_0, R],\ |y| \in (\lambda, R]. \tag{5.24}$$

Setting $m = \min_{\partial B_R(0)} w_{\lambda_0} > 0$ and since $-\Delta w_{\lambda_0} > 0$ in Σ_{λ_0},

$$w_{\lambda_0}(y) \geq m\frac{R^{n-2}}{|y|^{n-2}}, \text{ for } |y| \geq R.$$

Hence,

$$w_\lambda(y) \geq m\frac{R^{n-2}}{|y|^{n-2}} - (u_\lambda(y) - u_{\lambda_0}(y)), \text{ for } |y| \geq R. \tag{5.25}$$

By the uniform continuity of u on $\bar{B}_R(0)$, there exists $\epsilon \in (0, R - \lambda_0)$ such that for $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$,

$$\left|\lambda^{n-2} u\Big(\lambda^2 \frac{y}{|y|^2}\Big) - \lambda_0^{n-2} u\Big(\lambda_0^2 \frac{y}{|y|^2}\Big)\right| < \frac{mR}{2}, \text{ for } |y| \geq R.$$

From this and (5.25), we get

$$w_\lambda(y) > 0 \text{ for } \lambda \in [\lambda_0, \lambda_0 + \epsilon],\ |y| \geq R. \tag{5.26}$$

However, estimates (5.24) and (5.26) contradict the definition of λ_0. This proves the claim.

Step 2: We claim that if $\lambda_0(x_0) = \infty$ for some $x_0 \in \mathbb{R}^n$, then $\lambda_0(x) = \infty$ for all $x \in \mathbb{R}^n$.

Observe that, by definition,

$$u_{x_0,\lambda}(y) \leq u(y), \quad \text{for all } \lambda > 0,\ |y - x_0| \geq \lambda.$$

Thus,

$$\lim_{|y|\to\infty} |y|^{2-n} u(y) = \infty.$$

Assume that $\lambda_0(x) = \infty$ for some $x \in \mathbb{R}^n$. Then by Step 1,

$$\lim_{|y|\to\infty} |y|^{n-2} u(y) = \lim_{|y|\to\infty} |y|^{n-2} u_{x,\lambda_0(x)}(y) = \lambda_0(x)^{n-2} u(x) < \infty,$$

and we arrive at a contradiction.

Step 3: We claim $\lambda_0(x) < \infty$ for all $x \in \mathbb{R}^n$.

To see this, note that if $\lambda_0(x_0) = \infty$ for some point $x_0 \in \mathbb{R}^n$, then Step 2 ensures $\lambda_0(x) = \infty$ for all $x \in \mathbb{R}^n$. Lemma 5.2 then implies that $u \equiv$ *constant*. Since u is assumed to be positive and we have shown it is necessarily constant, we arrive at a contradiction.

Step 4: We are now ready to complete the proof of the theorem in the critical case. From the previous steps, for each $x \in \mathbb{R}^n$ it follows that $\lambda_0(x) < \infty$ and $u_{x,\lambda_0(x)} \equiv u$ in $\mathbb{R}^n \backslash \{x\}$. From Lemma 5.1, there are $a, d > 0$ and some $x_0 \in \mathbb{R}^n$ such that

$$u(x) = \Big(\frac{a}{d + |x - x_0|^2}\Big)^{\frac{n-2}{2}},\ x \in \mathbb{R}^n.$$

This proves the result in the critical case.

Subcritical case: Let $p < (n+2)/(n-2)$ and suppose u is a non-negative solution of (5.1). The proof in this case is similar to the critical case. Namely, due to the Kelvin transform, we can show that $\lambda_0(x_0) = \infty$ for some $x_0 \in \mathbb{R}^n$. As before, this implies that $\lambda_0(x) = \infty$ for each $x \in \mathbb{R}^n$. Then, by Lemma 5.2, $u \equiv$ *constant* and so $u \equiv 0$. This completes the proof. □

CHAPTER 6

Concentration and Non-compactness of Critical Sobolev Embeddings

6.1 Introduction

In this chapter, we explore the breakdown of the compactness of the injection

$$W^{1,p}(U) \hookrightarrow L^q(U)$$

where $1/q = 1/p - 1/n$ (see the appendix A for the statements and proofs of the Sobolev inequalities and embeddings). A closely related and important issue is when weak compactness fails to imply strong compactness. We have already encountered problems from the calculus of variations in which we recover the strong compactness of a minimizing sequence from its weak compactness by exploiting the coercivity and the weak lower semi-continuity of the functional undergoing minimization. Here we focus on the case when this compactness issue arises from a concentration phenomena due to an inherent scaling invariance in the problem. The approach we introduce to regain strong convergence (concentration compactness) is to show that concentration only occurs in a small or negligible set. We follow the notes of L. C. Evans [8], but we also refer the reader to P. L. Lions [22, 23]

To illustrate the key points, let us discuss the possibility that a sequence $f_k \rightharpoonup f$ weakly in $L^q(U)$ fails to converge strongly in $L^q(U)$. In addition to assuming weak convergence, let us also assume pointwise convergence almost everywhere, $f_k \longrightarrow f$ a.e. in U. This ensures that no wild oscillations may occur, which itself is another potential culprit responsible for the failure of strong convergence. Even this additional assumption, however, does not guarantee strong convergence due to a possible concentration of mass onto a negligible set. Namely, the obstruction is that the mass $|f_k - f|^q$ may somehow coalesce onto a set with Lebesgue measure zero. It is this concentration of mass onto a negligible or small set that allows us to overcome the breakdown of strong convergence in certain problems; for example, when proving the existence of extremal functions to sharp geometric inequalities (e.g., isoperimetric, Sobolev and Hardy-Littlewood-Sobolev inequalities).

The model example we focus on is the problem of proving the existence of extremal functions to a sharp Sobolev inequality; namely, the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, where $n > p = 2$ and $q = 2^* := 2n/(n-2)$. In particular, we first give a simple characterization of the non-compactness of the Sobolev embedding in terms of concentration. Then, we use this characterization and the concentration compactness principle to recover the strong compactness of minimizing sequences via translations and dilations to obtain an extremal function to the sharp Sobolev inequality. Prior to stating our main results, we review some terminology and basic theorems but we omit their proofs.

Theorem 6.1. *Let $U \subset \mathbb{R}^n$ be a bounded open subset, $1 \leq q < \infty$, and assume $f_k \rightharpoonup f$ in $L^q(U)$. Then*

(a) $\{f_k\}_{k=1}^{\infty}$ is bounded in $L^q(U)$ and

(b) $\|f\|_{L^q(U)} \leq \liminf_{k\to\infty} \|f_k\|_{L^q(U)}$.

(c) Refinement of Part (b): If $1 < q < \infty$, $f_k \rightharpoonup f$ in $L^q(U)$ and $\|f\|_{L^q(U)} = \lim_{k\to\infty} \|f_k\|_{L^q(U)}$, then

$$f_k \longrightarrow f \text{ strongly in } L^q(U).$$

Recall the following special case of the Banach-Alaoglu theorem.

Theorem 6.2. *Assume $1 < q < \infty$. If the sequence $\{f_k\}_{k=1}^{\infty}$ is bounded in $L^q(U)$, then it is weakly precompact in $L^q(U)$. That is, there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty}$ and a function $f \in L^q(U)$ such that $f_{k_j} \rightharpoonup f$ in $L^q(U)$.*

Remark 6.1. *The previous result holds in the case $q = \infty$ but the convergence of the subsequence in $L^\infty(U)$ is understood in the weak star sense, since $U \subseteq \mathbb{R}^n$ is σ-finite and $L^\infty(U)$ is isometrically isomorphic to the dual space $L^1(U)^*$. Namely, we treat sequences in $L^\infty(U)$ as sequences of bounded linear functionals on $L^1(U)$. The weak compactness in the case $q = 1$ is obviously false. To circumvent this issue, the Riesz Representation Theorem indicates that it is natural to consider $L^1(U)$ as a subset of $\mathcal{M}(U)$, the space of signed finite Radon measures on U.*

Definition 6.1. *A sequence $\{\mu_k\}_{k=1}^{\infty} \subset \mathcal{M}(U)$* **converges weakly** *to $\mu \in \mathcal{M}(U)$, written as*

$$\mu_k \rightharpoonup \mu \text{ in } \mathcal{M}(U),$$

provided that

$$\int_U g \, d\mu_k \longrightarrow \int_U g \, d\mu \text{ as } k \longrightarrow \infty$$

for each $g \in C_c(U)$.

Theorem 6.3. *Assume $\mu_k \rightharpoonup \mu$ weakly in $\mathcal{M}(U)$. Then*

$$\limsup_{k\to\infty} \mu_k(K) \le \mu(K)$$

for each compact set $K \subset U$, and

$$\mu(V) \le \liminf_{k\to\infty} \mu_k(V)$$

for each open set $V \subset U$.

Theorem 6.4 (Weak Compactness for Measures)**.** *Assume the sequence $\{\mu_k\}_{k=1}^\infty$ is bounded in $\mathcal{M}(U)$. Then there exists a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ and a measure μ in $\mathcal{M}(U)$ such that $\mu_{k_j} \rightharpoonup \mu$ in $\mathcal{M}(U)$.*

Remark 6.2. *We extend the terminology above to the Sobolev space $W^{1,q}(U)$, $1 \le q < \infty$, by saying that $f_k \rightharpoonup f$ weakly in $W^{1,q}(U)$ whenever $f_k \rightharpoonup f$ in $L^q(U)$ and $Df_k \rightharpoonup Df$ in $L^q(U;\mathbb{R}^n)$.*

Theorem 6.5 (Compactness for Measures)**.** *Assume the sequence $\{\mu_k\}_{k=1}^\infty$ is bounded in $\mathcal{M}(U)$. Then $\{\mu_k\}_{k=1}^\infty$ is precompact in $W^{-1,q}(U)$ for each $1 \le q < 1^*$.*

We will need the following refinement of Fatou's lemma (see Lemma A.1) due to Brezis and Lieb.

Theorem 6.6 (Refined Fatou)**.** *Let $1 \le q < \infty$ and assume $f_k \rightharpoonup f$ weakly in $L^q(U)$ and $f_k \longrightarrow f$ a.e. in U. Then*

$$\lim_{k\to\infty} \left(\|f_k\|^q_{L^q(U)} - \|f_k - f\|^q_{L^q(U)} \right) = \|f\|^q_{L^q(U)}.$$

To better understand how weak convergence in $L^q(U)$ fails to imply strong convergence in $L^q(U)$, we assume

$$f_k \rightharpoonup f \text{ in } L^q(U), \tag{6.1}$$

and consider the measures

$$\theta_k = |f_k - f|^q \text{ for } k = 1, 2, 3, \ldots.$$

Thus, each Radon-Nikodym derivative $\theta_k(E) = \int_E |f_k - f|^q\, dx$ controls how close f_k is to f in the L^q-norm restricted to the Borel set $E \subset U$. Now for each Borel set $E \subset U$, we call

$$\theta(E) = \limsup_{k\longrightarrow\infty} \int_E |f_k - f|^q\, dx$$

the **reduced defect measure** associated with the weak convergence (6.1) (in addition, we can show that θ is a finitely-additive outer measure). Then, we may characterize strong convergence in terms of this reduced defect measure.

Proposition 6.1. *Let $1 < q < \infty$, $E, F \subset U$ are Borel sets and suppose $f_k \rightharpoonup f$ in $L^q(U)$. Then*

(a) $\mu((E\backslash F) \cup (F\backslash E)) = 0$ implies $\theta(E) = \theta(F)$, where μ is the n-dimensional Lebesgue measure; and

(b) $f_k \longrightarrow f$ strongly in $L^q(E)$ if and only if $\theta(E) = 0$.

Therefore, we see the failure of strong convergence occurs if and only if $\theta(U) > 0$. As already alluded to earlier, our hope in this situation is for concentration to occur, i.e., θ is only non-trivial in a small or thin subset. Measuring smallness or thinness of sets can be delicate and we shall do so via p-capacities and Hausdorff measures. We define these now but the notion of p-capacity was already discussed when introducing the Wiener criterion and the Perron method (see (2.25) in Chapter 2).

Definition 6.2. *If $s \in [0, \infty)$, $\delta \in (0, \infty]$, we define the* **s-dimensional Hausdorff premeasure** *$H^s_\delta(A)$ by*

$$H^s_\delta(A) = \inf\Big\{ \sum_{j=1}^{\infty} \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)} \Big(\frac{diam\, C_j}{2}\Big)^s \,\Big|\, A \subset \bigcup_{j=1}^{\infty} C_j,\ diam\, C_j \leq \delta \Big\}.$$

for each subset $A \subset \mathbb{R}^n$. Then the **s-dimensional Hausdorff measure** *H^s is given by*

$$H^s(A) := \lim_{\delta \longrightarrow 0} H^s_\delta(A) = \sup_{\delta > 0} H^s_\delta(A) \text{ for each } A \subset \mathbb{R}^n.$$

If $1 \leq p < n$, we define the **p-capacity** *Cap_p by*

$$Cap_p(A) = \inf\Big\{ \int_{\mathbb{R}^n} |Df|^p \, dx \,\Big|\, f \in L^{p^*}(\mathbb{R}^n),\ Df \in L^p(\mathbb{R}^n),\ A \subseteq interior\{f \geq 1\} \Big\}$$

for each $A \subseteq \mathbb{R}^n$.

Definition 6.3. *We say θ is* **concentrated on a set of p-capacity zero** *if there exist open sets $\{V_i\}_{i=1}^{\infty}$ in U such that*

$$\theta(U\backslash V_i) = 0 \text{ for } i = 1, 2, 3, \ldots, \text{ and } Cap_p(V_i) \longrightarrow 0.$$

We say θ is **concentrated on a set of Hausdorff H^s-measure zero** *if there exist open sets $\{V_i\}_{i=1}^{\infty}$ in U and a sequence $\{\delta_i\}_{i=1}^{\infty}$ in $(0, \infty)$ such that*

$$\theta(U\backslash V_i) = 0 \text{ for } i = 1, 2, 3, \ldots,\ \delta_i \longrightarrow 0, \text{ and } H^s_{\delta_i}(V_i) \longrightarrow 0.$$

Roughly speaking, the last two definitions describe when θ concentrates on the set $V = \bigcap_{i=1}^{\infty} V_i$ with either $Cap_p(V) = 0$ or $H^s(V) = 0$. As θ is only finitely subadditive, we cannot generally deduce from this that $\theta(U\backslash V) = \theta(V^c) = 0$. For example, let $U = (0, 1)$, $f \equiv 0$ and

$$f_k(x) = \begin{cases} k, & \text{if } \frac{1}{2} - \frac{1}{2k} \leq x \leq \frac{1}{2} + \frac{1}{2k}, \\ 0, & \text{otherwise.} \end{cases}$$

Then θ is concentrated on $V = \{1/2\}$, $\theta(E^c) = 0$ for each open set E containing V but $\theta(V^c) = 1$.

6.2 Concentration and Sobolev Inequalities

Let C_2 be the best constant in the Gagliardo-Nirenberg-Sobolev inequality in this case (see A.11 in the appendix A). There holds the following.

Theorem 6.7. *Assume that* $n \geq 3$,

$$f_k \longrightarrow f \text{ strongly in } L^2_{loc}(\mathbb{R}^n), \quad Df_k \rightharpoonup Df \text{ in } L^2(\mathbb{R}^n;\mathbb{R}^n).$$

Suppose further that

$$|Df_k|^2 \rightharpoonup \mu \text{ in } \mathcal{M}(\mathbb{R}^n), \quad |f_k|^{2^*} \rightharpoonup \nu \text{ in } \mathcal{M}(\mathbb{R}^n).$$

(a) Then there exists an at most countable index set J*, distinct points* $\{x_j\}_{j\in J} \subset \mathbb{R}^n$*, and non-negative weights* $\{\mu_j, \nu_j\}_{j\in J}$ *such that*

$$\nu = |f|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j}, \quad \mu \geq |Df|^2 + \sum_{j\in J} \mu_j \delta_{x_j}. \tag{6.2}$$

(b) Furthermore,

$$\nu_j \leq C_2^{2^*} \mu_j^{2^*/2} \ (j \in J). \tag{6.3}$$

(c) If $f \equiv 0$ *and*

$$\nu(\mathbb{R}^n)^{1/2^*} \geq C_2 \mu(\mathbb{R}^n)^{1/2},$$

then ν *is concentrated at a single point.*

Proof. **Step 1:** Assume first that $f \equiv 0$. Choosing $\varphi \in C_c^\infty(\mathbb{R}^n)$, from (A.11) we deduce that

$$\Big(\int_{\mathbb{R}^n} |\varphi f_k|^{2^*}\,dx\Big)^{\frac{1}{2^*}} \leq C_2 \Big(\int_{\mathbb{R}^n} |D(\varphi f_k)|^2\,dx\Big)^{\frac{1}{2}}.$$

Since $f_k \longrightarrow f \equiv 0$ strongly in $L^2_{loc}(\mathbb{R}^n)$, we obtain

$$\Big(\int_{\mathbb{R}^n} |\varphi|^{2^*}\,d\nu\Big)^{\frac{1}{2^*}} \leq C_2 \Big(\int_{\mathbb{R}^n} |\varphi|^2\,d\mu\Big)^{\frac{1}{2}}. \tag{6.4}$$

So by approximation, we have

$$\nu(E)^{1/2^*} \leq C_2 \mu(E)^{1/2} \tag{6.5}$$

where $E \subset \mathbb{R}^n$ is any Borel set. Now since μ is a finite measure, the set

$$D := \{x \in \mathbb{R}^n \mid \mu(\{x\}) > 0\}$$

is at most countable. Thus, we can write $D = \{x_j\}_{j\in J}$, $\mu_j := \mu(\{x_j\})$ $(j \in J)$ so that

$$\mu \geq \sum_{j\in J} \mu_j \delta_{x_j}.$$

From (6.5) and the theory of symmetric derivatives of Radon measures (see Federer), we conclude that $\nu \ll \mu$ and so for each Borel set E,

$$\nu(E) = \int_E D_\mu \nu \, d\mu \tag{6.6}$$

where

$$D_\mu \nu(x) := \lim_{r\to 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}. \tag{6.7}$$

But (6.5) implies

$$\frac{\nu(B_r(x))}{\mu(B_r(x))} \leq C_2^{2^*} \mu(B_r(x))^{2/(n-2)}, \tag{6.8}$$

provided that $\mu(B_r(x)) \neq 0$. Thus, we infer

$$D_\mu \nu = 0 \quad \mu - a.e. \text{ on } \mathbb{R}^n \backslash D. \tag{6.9}$$

Now define $\nu_j := D_\mu \nu(x_j)\mu_j$. Then (6.6)-(6.9) imply parts (a) and (b) of the theorem (for $f \equiv 0$).

Step 2: Next, assume the hypotheses of assertion (c) in the theorem. Then (6.5) gives

$$\nu(\mathbb{R}^n)^{1/2^*} = C_2 \mu(\mathbb{R}^n)^{1/2}.$$

Since (6.4) ensures that

$$\Big(\int_{\mathbb{R}^n} |\varphi|^{2^*} \, d\nu\Big)^{\frac{1}{2^*}} \leq C_2 \mu(\mathbb{R}^n)^{\frac{1}{n}} \Big(\int_{\mathbb{R}^n} |\varphi|^2 \, d\mu\Big)^{\frac{1}{2}},$$

we deduce that $\nu = C_2^{2^*} \mu(\mathbb{R}^n)^{2/(n-2)} \mu$. Consequently, (6.4) reads

$$\Big(\int_{\mathbb{R}^n} |\varphi|^{2^*} \, d\nu\Big)^{\frac{1}{2^*}} \leq C_2 \nu(\mathbb{R}^n)^{\frac{1}{n}} \Big(\int_{\mathbb{R}^n} |\varphi|^2 \, d\nu\Big)^{\frac{1}{2}},$$

and so $\nu(E)^{1/2^*} \nu(\mathbb{R}^n)^{1/n} \leq \nu(E)^{1/2}$ for each Borel set E. This cannot happen if ν is concentrated at more than one point.

Step 3: Now assume $f \not\equiv 0$ and write $g_k := f_k - f$. The calculations in the Steps 1 and 2 apply to $\{g_k\}_{k=1}^\infty$ as well. Moreover, there holds

$$|Dg_k|^2 = |Df_k|^2 - 2Df_k \cdot Df + |Df|^2 \rightharpoonup \mu - |Df|^2 \text{ in } \mathcal{M}(\mathbb{R}^n),$$

and Theorem 6.6 implies $|g_k|^{2^*} \rightharpoonup \nu - |f|^{2^*}$ in $\mathcal{M}(\mathbb{R}^n)$. This completes the proof. and

□

6.3 Minimizers for Critical Sobolev Inequalities

Let $n \geq 3$ and consider the problem of minimizing the functional

$$I[w] = \int_{\mathbb{R}^n} |Dw|^2 \, dx, \tag{6.10}$$

over the admissible set

$$M := \{w \in L^{2^*}(\mathbb{R}^n) \,|\, \|w\|_{L^{2^*}(\mathbb{R}^n)} = 1, \; Dw \in L^2(\mathbb{R}^n; \mathbb{R}^n)\}.$$

Notice carefully that

$$I := \inf_{w \in M} I[w] = C_2^{-2}.$$

Our goal is to show that this infimum is indeed obtained by a suitable minimizer. On a related note, we may also consider the same minimization problem but on an arbitrary domain U with functional

$$I_U[w] = \int_U |Dw|^2 \, dx$$

undergoing minimization over

$$M_U := \{w \in L^{2^*}(U) \,|\, \|w\|_{L^{2^*}(U)} = 1, \; Dw \in L^2(U; \mathbb{R}^n)\}.$$

Interestingly enough, the infimum here is also given by the best constant in the Gagliardo-Nirenberg-Sobolev inequality, i.e.,

$$\min_{w \in M_U} I_U[w] = I = C_2^{-2},$$

but the minimum is not achieved for $U \neq \mathbb{R}^n$ (see Theorem 6.2 below for a proof). In other words, the best constant in the sharp Sobolev inequality does not depend on the domain and the culprit responsible for this is the scaling invariance

$$u(x) \mapsto u_R(x) := R^{n/2^*} u(Rx) = R^{(n-2)/2} u(Rx), \; R > 0, \tag{6.11}$$

with respect to the norms in the Sobolev inequality. For instance, fix $u \in H_0^1(B_1(0))$ satisfying $\|u_R\|_{L^{2^*}(B_R(0))} = \|u\|_{L^{2^*}(B_1(0))} = 1$, but then $u_R \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ as $R \longrightarrow \infty$. Therefore, relative strong compactness of minimizing sequences is not expected to hold. What ultimately saves us is the actions of rescaling and translations, which can recover the relative compactness of minimizing sequences.

Remark 6.3. *(a) Recall that the method of moving planes indicates that the critical points of the functional I, which includes its minimizers, are essentially unique. Namely, all critical points must admit the form*

$$u_{\varepsilon,x_0}(x) = c(n)\left(\frac{\varepsilon}{\varepsilon^2 + |x - x_0|^2}\right)^{\frac{n-2}{2}} \tag{6.12}$$

for some $\varepsilon > 0$ and some point $x_0 \in \mathbb{R}^n$.

(b) *The classification of critical points in (a) also illustrates the concentration property which occurs in the critical Sobolev inequality. Indeed, upon normalizing, there holds*

$$C\|u_{\varepsilon,x_0}\|_{H^1(\mathbb{R}^n)} = \|u_{\varepsilon,x_0}\|_{L^{2^*}(\mathbb{R}^n)} = 1$$

so that the sequence $\{u_{\varepsilon,x_0}\}_{\varepsilon>0}$ *is bounded in these norms. However, as* $\varepsilon \longrightarrow 0$*, we have that*

$$\begin{cases} u_{\varepsilon,x_0}(x) \longrightarrow 0 & \text{for } x \neq x_0, \\ u_{\varepsilon,x_0}(x) = 1/\varepsilon^{(n-2)/2} \longrightarrow \infty & \text{for } x = x_0. \end{cases}$$

Now we choose a minimizing sequence $\{u_k\}_{k=1}^{\infty} \subset M$ with

$$I[u_k] \longrightarrow I. \tag{6.13}$$

We may assume $Du_k \rightharpoonup Du$ in $L^2(\mathbb{R}^n;\mathbb{R}^n)$ and $u_k \rightharpoonup u$ in $L^{2^*}(\mathbb{R}^n)$. Recall from Chapter 2 that

$$I[u] \leq \liminf_{k\to\infty} I[u_k] = \inf_{w\in M} I[w].$$

Hence, u is a minimizer as long as $u \in M$. Now, since we have

$$\|u\|_{L^{2^*}(\mathbb{R}^n)} \leq 1, \tag{6.14}$$

what is only left to verify is if $\|u\|_{L^{2^*}(\mathbb{R}^n)} = 1$. Once we verify this, we are done. Before we state and prove the main result, for $v \in M$, $y \in \mathbb{R}^n$ and $s > 0$, we define the rescaled function

$$v^{y,s}(x) := s^{-\frac{n-2}{2}} v\Big(\frac{x-y}{s}\Big) \quad (x \in \mathbb{R}^n).$$

Theorem 6.8. *Let* $\{u_k\}_{k=1}^{\infty} \subset M$ *satisfy* (6.13)*. Then there exist translations* $\{y_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ *and dilations* $\{s_k\}_{k=1}^{\infty} \subset (0,\infty)$ *such that the rescaled family* $\{u_k^{y_k,s_k}\}_{k=1}^{\infty} \subset M$ *is strongly precompact in* $L^{2^*}(\mathbb{R}^n)$*. In particular there exists a minimizer* $u \in M$ *of the functional* I*.*

Sketch of Proof. We outline the proof in five main steps.
Step 1: Define the Lévy concentration functions

$$Q_k(t) := \sup_{y\in\mathbb{R}^n} \int_{B_t(y)} |u_k|^{2^*}\,dx \quad (t > 0, k = 1,2,3,\ldots).$$

Then $Q_k^{y,s}(t) = Q_k^{y,1}(t/s)$ where $Q_k^{y,s}$ is the concentration function of $u_k^{y,s}$. The fact that

$$\lim_{t\to\infty} Q_k(t) = 1$$

ensures we can choose dilations $\{s_k\}_{k=1}^{\infty}$ such that

$$Q_k^{y,s_k}(1) = 1/2 \text{ for all } y \in \mathbb{R}^n,\ k = 1,2,3,\ldots.$$

Then this allows us to select translations $\{y_k\}_{k=1}^{\infty}$ so that the measures, $\nu_k^{y_k,s_k} = |u_k^{y_k,s_k}|^{2^*}$ $(k = 1, 2, 3 \ldots)$, are tight in $\mathcal{M}(\mathbb{R}^n)$.

Step 2: To simplify notation, we assume the dilations and translations of step one were unnecessary and so $Q_k(1) = 1/2$ $(k = 1, 2, 3, \ldots)$ and the measures $\{\nu_k\}_{k=1}^{\infty}$ are tight. Thus, passing to a subsequence, if necessary, we may assume

$$\nu_k \rightharpoonup \nu \text{ in } \mathcal{M}(\mathbb{R}^n), \quad \nu(\mathbb{R}^n) = 1. \tag{6.15}$$

We may also assume that

$$\mu_k \rightharpoonup \mu \text{ in } \mathcal{M}(\mathbb{R}^n) \tag{6.16}$$

for $\mu_k := |Du_k|^2$ $(k = 1, 2, 3, \ldots)$.

Step 3: We claim that $u \not\equiv 0$.

Assume the contrary. By noting that $\mu_k(\mathbb{R}^n) \longrightarrow I$, $\mu(\mathbb{R}^n) \leq I = C_2^{-2}$, and (6.15), we use part (c) of Theorem 6.7 to get that ν is concentrated at a single point $x_0 \in \mathbb{R}^n$. From this we deduce the contradiction

$$\frac{1}{2} = Q_k(1) \geq \int_{B_1(x_0)} |u_k|^{2^*} \, dx \longrightarrow 1.$$

Step 4: We claim that $u \in M$.

Assume otherwise, i.e., assume that $\|u\|_{L^{2^*}(\mathbb{R}^n)}^{2^*} = \lambda \in (0, 1)$. Setting

$$M_\lambda := \{w \in L^{2^*}(\mathbb{R}^n) \mid \|w\|_{L^{2^*}(\mathbb{R}^n)} = \lambda, \, Dw \in L^2(\mathbb{R}^n; \mathbb{R}^n)\},$$

we write

$$I_\lambda := \inf_{w \in M_\lambda} I[w].$$

Then $I_\lambda = \lambda^{2/2^*} I$.

Step 5: According to (a) and (b) of Theorem 6.7, we have

$$\nu = |u|^{2^*} + \sum_{j \in J} \nu_j \delta_{x_j}, \quad \mu \geq |Du|^2 + \sum_{j \in J} \mu_j \delta_{x_j}$$

for some countable set of points $\{x_j\}_{j \in J}$ and positive weights $\{\mu_j, \nu_j\}_{j \in J}$, satisfying

$$\lambda + \sum_{j \in J} \nu_j = 1, \quad \mu_j \geq \nu_j^{2/2^*} I \ (j \in J).$$

Hence, we arrive at the contradiction

$$\begin{aligned} I \geq \mu(\mathbb{R}^n) &\geq \int_{\mathbb{R}^n} |Du|^2 \, dx + \sum_{j \in J} \mu_j \\ &\geq I_\lambda + \sum_{j \in J} \mu_j \geq \Big(\lambda^{2/2^*} + \sum_{j \in J} \nu_j^{2/2^*}\Big) I \\ &> I, \end{aligned}$$

and this completes the proof. □

Remark 6.4. *Roughly speaking, Steps 3 to 5 in the proof show that vanishing and dichotomy in the principle of concentration compactness do not occur and therefore, compactness must hold (see Proposition 2.2). Step 5, in particular, shows that if a portion of the mass concentrates, our minimization problem splits into two parts, the sum of whose energies strictly exceeds the energy were splitting not to occur.*

6.4 A Sharp Sobolev Inequality

We are now in a position to combine our previous results to give a complete proof of the sharp Sobolev inequality

$$C_*^{-1}\|u\|^2_{L^{2^*}(U)} \leq \|Du\|^2_{L^2(U)} \text{ for every } u \in H^1_0(U), \tag{6.17}$$

where $n \geq 3$ is an integer, $2^* = 2n/(n-2)$ is the critical Sobolev exponent, U is an open domain, i.e., an open connected subset of $\mathbb{R}^n$, and the sharp constant $C_* = C_*(n)$ only depends on the dimension n and is explicitly given by

$$\begin{aligned}\frac{1}{C_*} &= \frac{n(n-2)}{4}|\mathbb{S}^n|^{2/n} = \frac{n(n-2)}{4}\Big(\frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}\Big)^{2/n} \\ &= n(n-2)\pi\Big(\frac{\Gamma(n/2)}{\Gamma(n)}\Big)^{2/n},\end{aligned} \tag{6.18}$$

where $\Gamma(\cdot)$ is the gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \quad \text{for } x > 0. \tag{6.19}$$

First, let us remark why the identities in (6.18) hold. Recall

$$|\mathbb{S}^n| = \omega_{n+1} = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$$

and thus to verify the last equality in (6.18), it suffices to show

$$\frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)} = (4\pi)^{n/2}\frac{\Gamma(n/2)}{\Gamma(n)}.$$

Indeed, this is equivalent to

$$2^{n-1}\Gamma(n/2)\Gamma((n+1)/2) = \pi^{1/2}\Gamma(n),$$

which in turn is equivalent to

$$\Gamma(n/2)\Gamma((n+1)/2) = 2^{1-n}\pi^{1/2}\Gamma(n).$$

Now the last equality follows by setting $x = n/2$ in the following gamma function identity

$$\Gamma(x)\Gamma\Big(\frac{2x+1}{2}\Big) = 2^{1-2x}\pi^{1/2}\Gamma(2x), \text{ for any } x > 0. \tag{6.20}$$

We mention another useful identity which we will use in computing the best constant in (6.18). Namely, we have

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \text{ for } x, y \in (0,\infty), \tag{6.21}$$

where $B(x,y) : (0,\infty) \times (0,\infty) \longrightarrow \mathbb{R}$ is the beta function defined by

$$B(x,y) = \int_0^1 r^{x-1}(1-r)^{y-1}\, dr.$$

As in the previous section, set

$$I_U[w] = \int_U |Dw|^2\, dx,$$

but here we denote

$$S(U) = \inf\{I_U(w) \,|\, w \in H_0^1(U), \|w\|_{L^{2^*}(U)} = 1\}$$

and $I = S(\mathbb{R}^n)$.

Notice that the sharp constant depends only on the spatial dimension rather than the domain itself. This further suggests that $S(U)$ should be independent of U and therefore remains constant over all domains. In other words, $S(U)$ is not attained for proper domains. We prove this in the next proposition, but we already know this to be true in the case of bounded star-shaped domains by the Rellich-Pohozaev identity (see Proposition 2.3). The next proposition is more general as it precisely states that the minimum is never attained on any domain unless $U = \mathbb{R}^n$. The key to proving this exploits the rigidity (scaling and translation invariance) of Sobolev inequalities in the whole space as discussed in (6.11). We then prove that if $U = \mathbb{R}^n$, the best constant is attained and the minimizer, as a result of the method of moving planes, is uniquely given by standard bubble functions.

Proposition 6.2. *Let $U \subseteq \mathbb{R}^n$ be an open somain.*

(a) The best constant in the Sobolev embedding $H_0^1(U) \hookrightarrow L^{2^*}(U)$ is independent of U; that is, $S(U_1) = S(U_2)$ for any open sets U_1 and U_2 in $\mathbb{R}^n$.

(b) If U is a proper domain of $\mathbb{R}^n$, then $S(U)$ is never attained.

Proof. **Part (a).** Suppose U_1 and U_2 are any open sets in $\mathbb{R}^n$. Since $S(U) = S(x_0 + U)$ for any $x_0 \in \mathbb{R}^n$, we may assume that 0 belongs to $U_1 \cap U_2$. Denote the rescaled function $w^R(x) = w(Rx)$. Choose $\varepsilon > 0$ and a non-trivial $u_1 \in H_0^1(U_1)$ such that $I_{U_1}(u_1) < S(U_1) + \varepsilon$. Define the extension function

$$\bar{u}_1(x) = \begin{cases} u_1(x), & \text{if } x \in U_1, \\ 0, & \text{if } x \notin U_1, \end{cases}$$

so that $\bar{u}_1$ belongs to $H_0^1(\mathbb{R}^n)$ and $supp(\bar{u}_1^R) \subset U_2$ provided $R > 0$ is large enough.

Now let u_2 be the restriction of $\bar{u}_1^R$ to U_2 so that u_2 is non-trivial and belongs to $H_0^1(U_2)$, and

$$S(U_2) \leq I_{U_2}(u_2) = I_{\mathbb{R}^n}(\bar{u}_1^R) = I_{\mathbb{R}^n}(\bar{u}_1) = I_{U_1}(u_1) < S(U_1) + \varepsilon.$$

We conclude that $S(U_2) \leq S(U_1)$. Similarly, exchanging the role of U_1 and U_2 and running the same argument as above, we also deduce that $S(U_1) \leq S(U_2)$. Therefore, $S(U_1) = S(U_2)$.

Part (b). Now suppose $U \neq \mathbb{R}^n$ and assume that $u \in H_0^1(U)$ is a non-negative minimizer of the functional $I_U(w)$ and thus $I_U(u) = S(U)$. Now set $v \equiv u$ in U and $v \equiv 0$ in U^c. Then $S(U) = S(\mathbb{R}^n)$ and v is a non-negative minimizer for $S(\mathbb{R}^n)$. Thus, recall that v must be a positive classical solution of

$$\Delta v + cv^{(n+2)/(n-2)} \text{ in } \mathbb{R}^n$$

for some constant $c > 0$. But this contradicts that v vanishes outside U. □

Theorem 6.9 (Sharp Sobolev)**.** *Let U be an open set in $\mathbb{R}^n$. For all $u \in H_0^1(U)$, there holds the inequality*

$$\frac{n(n-2)}{4}|\mathbb{S}^n|^{2/n}\Big(\int_U |u|^{\frac{2n}{n-2}}\, dx\Big)^{\frac{n-2}{n}} \leq \int_U |Du|^2\, dx. \tag{6.22}$$

If $U = \mathbb{R}^n$, then equality holds in the estimate if and only if $u(x)$ is a non-zero constant multiple of

$$\varphi_{\varepsilon,x_0}(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x - x_0|^2}\Big)^{\frac{n-2}{2}} \tag{6.23}$$

for some $\varepsilon > 0$ and some point x_0 in $\mathbb{R}^n$.

Proof. By the Gagliardo-Nirenberg-Sobolev inequality (A.11) and a basic density argument, there exists a positive constant C_*^{-1}, depending only on n, for which the inequality (6.17) holds in $H_0^1(U)$. By Proposition 6.2, it is enough to assume $U = \mathbb{R}^n$. Therefore, it remains to show that the best possible constant is given in (6.18) and the infimum in the associated variational problem is achieved by standard bubbles of the form (6.23).

Step 1 (Existence and classification of minimizers).

Theorem 6.8 ensures the existence of a non-trivial minimizer u to the variational problem $I = S(\mathbb{R}^n)$. Further, recall that u is a non-trivial minimizer if and only if it is proportional to any classical entire solution of the critical Lane-Emden equation

$$\Delta u + u^{\frac{n+2}{n-2}} = 0, u > 0, \text{ in } \mathbb{R}^n.$$

But Theorem 5.1 ensures every classical solution of this equation has the form (6.23).

Step 2 (Choosing a minimizer).

In this step, we choose the minimizer that we will use to compute the sharp constant. By invariance, we may choose $x_0 = 0$ and $\varepsilon = 1$ in $\varphi_{\varepsilon,x_0}(x)$ and so we take the minimizer to be the function

$$u(x) = c\Big(\frac{1}{1+|x|^2}\Big)^{\frac{n-2}{2}}, \tag{6.24}$$

where c is chosen so that $\|u\|_{L^{2^*}(\mathbb{R}^n)} = 1$, i.e., we want $c^{2n/(n-2)} \cdot A = 1$ where

$$\begin{aligned}
A &= \int_{\mathbb{R}^n} \Big[\Big(\frac{1}{1+|x|^2}\Big)^{(n-2)/2}\Big]^{2n/(n-2)} dx = \int_{\mathbb{R}^n} \Big(\frac{1}{1+|x|^2}\Big)^n dx \\
&= \int_0^\infty \int_{\partial B_t(0)} \Big(\frac{1}{1+t^2}\Big)^n dS\, dt = \int_0^\infty |\mathbb{S}^{n-1}| t^{n-1}(1+t^2)^{-n}\, dt \\
&= |\mathbb{S}^{n-1}| \int_0^\infty \frac{t^{n-2}}{(1+t^2)^{n-2}} \frac{t\, dt}{(1+t^2)^2} = \Big(\text{set } r = \frac{1}{1+t^2}, -\frac{dr}{2} = \frac{t\, dt}{(1+t^2)^2}\Big) \\
&= \frac{1}{2}|\mathbb{S}^{n-1}| \int_0^1 r^{\frac{n}{2}-1}(1-r)^{\frac{n}{2}-1}\, dr = \frac{1}{2}|\mathbb{S}^{n-1}| B(n/2, n/2) \\
&= \frac{1}{2}|\mathbb{S}^{n-1}| \frac{\Gamma(n/2)\Gamma(n/2)}{\Gamma(n)} = \frac{1}{2}\frac{2\pi^{n/2}}{\Gamma(n/2)}\frac{\Gamma(n/2)\Gamma(n/2)}{\Gamma(n)} = \pi^{n/2}\frac{\Gamma(n/2)}{\Gamma(n)}.
\end{aligned}$$

Hence,

$$c = \Big(\frac{\Gamma(n)}{\pi^{n/2}\Gamma(n/2)}\Big)^{\frac{n-2}{2n}}. \tag{6.25}$$

Step 3 (Compute the best constant).

If u is defined by (6.24) and (6.25), then an elementary but tedious calculation yields the best constant

$$
\begin{aligned}
C_*^{-1} = \; &= I_{\mathbb{R}^n}(u) = \int_{\mathbb{R}^n} |Du|^2\,dx = c^2(n-2)^2 \int_{\mathbb{R}^n} \Big(\frac{1}{1+|x|^2}\Big)^n |x|^2\,dx \\
&= c^2(n-2)^2|\mathbb{S}^{n-1}| \int_0^\infty t^{n-1}\frac{t^2}{(1+t^2)^n}\,dt \\
&= c^2(n-2)^2|\mathbb{S}^{n-1}| \int_0^\infty \frac{t^n}{(1+t^2)^{n-2}}\frac{t\,dt}{(1+t^2)^2} = \Big(\text{set } r = \frac{1}{1+t^2},\, -\frac{dr}{2} = \frac{t\,dt}{(1+t^2)^2}\Big) \\
&= c^2\frac{(n-2)^2}{2}|\mathbb{S}^{n-1}| \int_0^1 r^{\frac{n}{2}-2}(1-r)^{\frac{n}{2}}\,dr = c^2\frac{(n-2)^2}{2}|\mathbb{S}^{n-1}| B\Big(\frac{n}{2}-1, \frac{n}{2}+1\Big) \\
&= c^2\frac{(n-2)^2}{2}|\mathbb{S}^{n-1}|\frac{\Gamma(\frac{n}{2}-1)\Gamma(\frac{n}{2}+1)}{\Gamma(n)} \\
&= \Big(\frac{\Gamma(n)}{\pi^{n/2}\Gamma(n/2)}\Big)^{\frac{n-2}{n}}\frac{(n-2)^2}{2}\frac{2\pi^{n/2}}{\Gamma(n/2)}\frac{\Gamma(\frac{n}{2}-1)\Gamma(\frac{n}{2}+1)}{\Gamma(n)} \\
&= (n-2)^2\pi\Big(\frac{\Gamma(n/2)}{\Gamma(n)}\Big)^{2/n}\frac{\Gamma(\frac{n}{2}-1)\Gamma(\frac{n}{2}+1)}{\Gamma(n/2)\Gamma(n/2)} \\
&= (n-2)^2\pi\Big(\frac{\Gamma(n/2)}{\Gamma(n)}\Big)^{2/n}\frac{\Gamma(\frac{n}{2}-1)(n/2)\Gamma(n/2)}{(\frac{n}{2}-1)\Gamma(\frac{n}{2}-1)\Gamma(n/2)} \\
&= (n-2)^2\pi\Big(\frac{\Gamma(n/2)}{\Gamma(n)}\Big)^{2/n}\frac{n/2}{(n-2)/2} = n(n-2)\pi\Big(\frac{\Gamma(n/2)}{\Gamma(n)}\Big)^{2/n} = \frac{n(n-2)}{4}|\mathbb{S}^n|^{2/n}.
\end{aligned}
$$

□

CHAPTER 7

Other Related Problems Arising from Analysis and Geometry

This chapter surveys several fundamental mathematical problems in which elliptic PDEs play a prominent role in their solution. We shall briefly introduce and motivate each problem. We then give proofs of the corresponding results making sure to elucidate the essential ideas in the proofs while highlighting where the elliptic theory we have learned come into play.

7.1 Riemann Mapping and Uniformization Theorems

We start this chapter with one of the most remarkable and profound mathematical results of the 19th century, the Riemann Mapping theorem. This classical result asserts all "nice domains without holes", more precisely the open and simply connected proper subsets of the complex plane $\mathbb{C}$, are conformal to one another. The earliest statement of this result dates back to Riemann in his 1851 dissertation albeit with additional restrictions, e.g., he imposed sufficient regularity on the boundary of the domain. Riemann, however, offered an "incorrect" proof and although Koebe later received recognition for his subsequent proof of the result, the first complete proof was given earlier by W. Osgood in 1900 [25].

In keeping with our theme on elliptic PDEs, we present a proof of the Riemann mapping theorem in the spirit of Riemann's original ideas. Namely, Riemann's approach aimed to construct a conformal mapping from any simply connected domain onto the unit disk. This effectively reduces the problem to solving an elliptic boundary value problem. At the time, Riemann and his contemporaries lacked the modern analytical tools to properly tackle such a problem. That is, he attempted to solve the problem through variational arguments via Dirichlet's principle. However, the validity of this principle, in particular, on successfully minimizing the Dirichlet energy integral $\int_U |Du|^2\, dx$ over an appropriate class of functions was not yet settled, and his approach proved controversial among experts at that time.

Thanks to basic elliptic PDE theory, properly carrying out Riemann's original idea is now fairly routine. Rather than use Dirichlet's principle, however, we utilize the Perron method for solving the boundary value problem, since it is more effective in dealing with the boundary of an arbitrary simply connected domain.

Remark 7.1. *Riemann's original approach contrasts with the typical arguments that makes use of the theorems of Montel and Hurwitz, which nowadays has become the norm in many complex analysis textbooks. The reader is referred to the textbook [11], which provides a proof from both perspectives. In fact, our presentation is inspired by discussions from that textbook as well as the paper [15].*

Theorem 7.1 (Riemann Mapping)**.** *If U is a simply connected domain in $\mathbb{C}$, and $U \neq \mathbb{C}$, then U is confomally equivalent to the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.*

For the reader's convenience, some background definitions are in order. Let U, U_1 and U_2 be domains in the complex plane. We say a path $\gamma : [a, b] \longrightarrow U$ in U is closed if $\gamma(a) = \gamma(b)$, and we say it is simple if $\gamma(t_1) \neq \gamma(t_2)$ for all $t_1 \neq t_2$ in (a, b). Then U is said to be simply connected if each simple closed path in U can be deformed or contracted to a single point. For instance, any star-shaped domain, such as an open ball, is simply connected. Moreover, we say U_1 and U_2 are conformally equivalent if there is a conformal mapping $f : U_1 \longrightarrow U_2$ of one domain onto the other.

Recall a smooth complex-valued function $f(z)$ is conformal at z_0 if whenever γ_0 and γ_1 are two curves terminating at z_0 with non-zero tangents, then the curves $f \circ \gamma_0$ and $f \circ \gamma_1$ have non-zero tangents at $f(z_0)$ and the angle from $(f \circ \gamma_0)'(z_0)$ to $(f \circ \gamma_1)'(z_0)$ is the same as the angle from $\gamma_0'(z_0)$ to $\gamma_1'(z_0)$. Note, if $f(z)$ analytic at a point z_0 and $f'(z_0) \neq 0$, then $f(z)$ is conformal at z_0. Moreover, a conformal mapping of one domain U_1 onto another U_2 is a continuously differentiable function that is conformal at each point of U_1 and that maps U_1 one-to-one onto U_2.

We say a subset E of $\hat{\mathbb{C}}$ is a continuum if it is compact and connected and contains more than one point. For a continuum E, then its complement $E^c = \hat{\mathbb{C}} \backslash E$ is simply connected. Conversely, if E is compact and E^c is simply connected, and if E contains more than one point, then E is a continuum.

Theorem 7.1 easily generalizes to the uniformization theorem, which classifies all simply connected domains (more generally, Riemann surfaces) into three basic models. To state this classification result, we first define the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{+\infty\}$, which is the one-point compactification of the complex plane, sometimes referred to as the Riemann sphere. To motivate why we refer to it as a sphere, we recall the stereographic projection from the unit n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ onto the Euclidean space $\mathbb{R}^n$. More precisely, we let $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{+\infty\}$ denote the one-point compactification of $\mathbb{R}^n$, and we define $(s_1, s_2, \ldots, s_{n+1})$ by

$$s_i = \frac{2x_i}{1 + |x|^2} \ \text{ for } i = 1, 2, \ldots, n, \ \text{ and } \ s_{n+1} = \frac{1 - |x|^2}{1 + |x|^2} \ (x \in \mathbb{R}^n). \tag{7.1}$$

If $x = +\infty$, we set $s_i = 0$ for $i = 1, 2, \ldots, n$ and $s_{n+1} = -1$. Thus, we are identifying the "north pole" and "south pole" of the unit sphere with the origin $x = 0$ and $+\infty$, respectively. Moreover, it is easy to show $\sum_{i=1}^{n+1} s_i^2 = 1$. Thus, $\mathcal{S} : x \longrightarrow s$ is a mapping from $\hat{\mathbb{R}}^n$ onto $\mathbb{S}^n$. The inverse of this mapping (this inverse map, in particular, is what we typically call the stereographic projection of the unit n-sphere onto a plane) is given by

$$x_i = \frac{s_i}{1 + s_{n+1}} \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{7.2}$$

Remark 7.2. *If one situates the n-sphere to lie on the hyperplane $\mathbb{R}^n$ with its south pole touching the origin of the hyperplane, we may visualize the stereographic projection as the one-to-one correspondence between the points on the sphere and the hyperplane connected by the line emanating from the north pole.*

Evidently, the map $\mathcal{S}$ is a conformal mapping and thus so is its inverse. And, in the case of the extended complex plane, we may take $n = 2$ in the mapping $\mathcal{S} : \hat{\mathbb{R}}^2 \simeq \hat{\mathbb{C}} \longrightarrow \mathbb{S}^2$ to "lift" the extended complex plane onto the 2-sphere. Thus, we may view the extended complex plane as a sphere, i.e., we call $\hat{\mathbb{C}}$ the Riemann sphere.

The assumption $U \neq \mathbb{C}$ in Theorem 7.1 is necessary. According to the Liouville theorem for harmonic functions, the only bounded harmonic functions in the entire (complex) plane are constant maps. Therefore $\mathbb{C}$ cannot be mapped conformally onto any bounded domain. In the context of simply connected domains on the Riemann sphere, if such a domain U is not the Riemann sphere, we can always translate a point in the complement of U to $+\infty$ via a fractional linear transformation then apply Theorem 7.1. Therefore, U must either be conformal to the unit disk or the complex plane. Namely, we deduce a simplified version of the uniformization theorem: any simply connected domain can be classified into the three standard models: the open unit disk, the complex plane, or the Riemann sphere.

Corollary 7.1. *A simply connected domain in the Riemann sphere is either the Riemann sphere itself, or it is conformally equivalent to the complex plane, or it is conformally equivalent to the open unit disk.*

Proof of Theorem 7.1. Suppose U is a simply connected domain and $U^c = \hat{\mathbb{C}} \backslash U$ has a least two points. Clearly, we assume $U \not\equiv \mathbb{D}$. And recall that we can use the logarithm function composed with a fractional linear transformation to map U conformally onto a bounded domain. More precisely, if $\zeta_0, \zeta_1 \in \partial U$, we may set $f(z)$ to be an analytic branch of $\log((z - \zeta_0)/(z - \zeta_1))$ in U. If $w_0 = f(z_0)$ for some $z_0 \in U$, then the image $f(U)$ is contained in some ball $B_R(w_0)$ and $f(z)$ cannot assume any of the values in the set $B_R(w_0) + 2\pi i$ for any $z \in U$. So we deduce that $1/(f(z) - w_0 - 2\pi i)$ maps U conformally onto some bounded domain. So without loss of generality, we may assume U is bounded. We may also assume $0 \in U$ by translation.

For such a domain, we seek a function $u(z) : U \longrightarrow \mathbb{C}$ solving the Dirichlet problem for Laplace's equation,

$$\begin{cases} \Delta u(z) = 0 & \text{for } z \in U, \\ u = \log |\zeta| & \text{for } \zeta \in \partial U. \end{cases} \tag{7.3}$$

The existence of a unique harmonic function $u(z)$ satisfying (7.3) follows from the Perron method, provided each point on the boundary ∂U is a regular point (see Chapter 2 Remark 2.9 and Theorem 2.17). This is indeed the case for simply connected bounded domains as the following lemma indicates, which we state without proof.

Lemma 7.1. *Let $\zeta \in \partial U$. If ζ lies on a continuum in $\mathbb{C}\backslash U$, then every point of the boundary ∂U is regular. Consequently, if $\hat{\mathbb{C}}\backslash U$ consists of finitely-many continua, then every point of the boundary ∂U is regular.*

Now, for this harmonic function $u(z)$ solving (7.3), the Cauchy-Riemann equations yield its harmonic conjugate $v(z)$ in U. We then define the mapping

$$\varphi(z) = ze^{-(u(z)+iv(z))} \quad \text{for} \quad z \in U. \tag{7.4}$$

Obviously, $\varphi(z)$ is non-constant and analytic in U and $|\varphi(\zeta)| = 1$ on the boundary ∂U. The strong maximum principle ensures that $|\varphi(z)| < 1$ in U. This verifies φ maps U into the open unit disk. So, it only remains to show this mapping is onto and therefore a conformal mapping. The final step will make use of the so-called Argument Principle.

Lemma 7.2 (Argument Principle)**.** *Let D be a bounded domain of the complex plane with piecewise smooth boundary ∂D, and suppose $g(z)$ is a meromorphic function on D (i.e., $g(z)$ is analytic on D except possibly at isolated singularities each of which is a pole), that extends to be analytic on ∂D, such that $g(z) \neq 0$ for all $z \in \partial D$. Then*

$$\frac{1}{2\pi i}\int_{\partial D} d\log g(z) := \frac{1}{2\pi i}\int_{\partial D} \frac{g'(z)}{g(z)}\, dz = N_0 - N_\infty,$$

where N_0 is the number of zeros of $g(z)$ in D and N_∞ is the number of poles of $g(z)$ in D, counting multiplicities.

By our construction, $\varphi(z)$ has only one zero in U, a simple zero at $z = 0$. Now pick any point w in the open unit disk, i.e., $|w| < 1$, and consider the domain $\{z \in U \,:\, |\varphi(z)| < 1-\varepsilon\}$, where $|w| < 1-\varepsilon$ for some $0 < \varepsilon < 1$. We apply Lemma 7.2 to this domain and the function $g(z) = \varphi(z) - w$ to conclude that $\varphi(z) - w$ has exactly one zero in U. Hence, $\varphi(z)$ must map U conformally onto the open unit disk. □

Remark 7.3. *Let us clarify our application of Lemma 7.2 in the proof of Theorem 7.1. In general, suppose $\varphi(z)$ is a non-constant analytic function on some domain D, $z_0 \in D$, $w_0 = \varphi(z_0)$ and assume $\varphi(z) - w_0$ has a zero of order m at z_0. Since the zeros of $\varphi(z) - w_0$ are isolated, we can find $\rho > 0$ so that $\varphi(z) - w_0 \neq 0$ for $0 < |z - z_0| \leq \rho$. Pick $\delta > 0$ to satisfy $|\varphi(z) - w_0| \geq \delta$ for $|z - z_0| = \rho$. Then, the integral*

$$N(w) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{\varphi'(z)}{\varphi(z) - w}\, dz, \quad \textit{with} \quad |w - w_0| < \delta,$$

is defined, is an analytic function in the disk $|w - w_0| < \delta$*, and* $N(w)$ *is the number of zeros of* $\varphi(z) - w$ *in the disk* $|z - z_0| < \rho$*. Therefore,* $N(w)$ *is integer-valued and thus constant. Since* $N(w_0) = m$*, we obtain* $N(w) = m$ *for* $|w - w_0| < \delta$*. In other words,* $\varphi(z)$ *attains each value* w*, with* $|w - w_0| < \delta$*,* m *times in the disk* $|z - z_0| < \rho$*.*

In our proof above, we are defining $\varphi(z)$ *as in* (7.4) *and taking* $z_0 = w_0 = 0$*,* $m = 1$*,* $\delta = 1 - \varepsilon$*, and applying the Argument Principle accordingly.*

7.2 The Yamabe Problem

This section introduces the celebrated Yamabe problem from conformal geometry, and we discuss its history, motivation, then give a detailed outline of its solution. We shall try our best to give a relatively self-contained overview of the Yamabe problem giving the necessary background in Riemannian geometry and the theory of elliptic PDEs on manifolds along the way. We refer the reader to Section B in the appendices and the references therein for a basic introduction of Riemannian geometry, at least the material needed for our purposes here. The Yamabe problem is stated as follows.

Conjecture 1 (The Yamabe Problem)**.** *Given a closed Riemannian manifold* (M, g) *of dimension* $n \geq 3$*, there exists a metric conformal to* g *with constant scalar curvature.*

In other words, this conjecture asserts the existence of some real-valued function $f \in C^\infty(M)$ such that the conformal change of metric $\tilde{g} = e^{2f} g$ admits constant scalar curvature. By a closed manifold, we mean the manifold is compact and has no boundary. A simple example is the standard sphere with metric induced from the Euclidean metric.

The validity of the Yamabe problem indeed holds, but it took the combined efforts of several mathematicians nearly three decades to solve completely. Conjecture 1 was first raised by Hidehiko Yamabe in the 1950s, and its formulation was motivated by the fact its solution may offer an alternative approach to attacking the Poincaré conjecture via analysis rather than by topological means. It can also be viewed as a generalization of the two-dimensional setting, which was well understood by that time. More precisely, it generalizes the renowned uniformization theorem (we studied a special case of this in the previous section).

Theorem 7.2 (Uniformization Theorem for Riemann Surfaces)**.** *Each simply connected (Riemann) surface is conformally equivalent to either one of the following constant curvature models: (a) the open unit disk (hyperbolic space), (b) the complex plane (flat space) or (c) the Riemann sphere (compact space).*

Yamabe offered a proof of Conjecture 1 in 1960 [36], however, N. Trudinger found a gap in Yamabe's proof some years later. Although Trudinger managed to repair Yamabe's original proof, he did so under the extra assumption that a particular energy quantity of the manifold, now known as the Yamabe invariant, did not exceed a certain threshold [31]. T. Aubin improved Trudinger's result by identifying the energy threshold (it is closely-related to the sharp Sobolev constant) and extended the result to hold for locally conformally flat manifolds in dimension $n \geq 6$. Using a novel approach (and different ideas from before) and the Positive Mass Theorem, R. Schoen finally settled the remaining cases in [29].

Our notes here will attempt to give a concise outline of the proof of the Yamabe problem in the spirit of Yamabe, Trudinger and Aubin. For a complete treatment of the Yamabe problem and for the technical details on aspects we do not cover here carefully, please see the very nice paper of Lee and Parker [17], in addition to the aforementioned papers.

We let (M, g) denote a Riemannian manifold with metric tensor g and volume element dV_g, where $M = M^n$ is always taken to be a smooth and connected n-manifold. We denote by Ric_g and S_g the Ricci and scalar curvatures.

7.3 The Isoperimetric Inequality

We first introduce the so-called isoperimetric problem and the resulting sharp inequality in the lower dimensional setting. That is, let us consider any simply closed plane curve $\mathcal{C}$ with prescribed fixed perimeter L, and we let A denote the area enclosed by $\mathcal{C}$. A classical problem asks among all such possible $\mathcal{C}$ with fixed length L, does there exist one that gives maximal area? Indeed, it turns out the circle of length L uniquely (up to translations) maximizes enclosed area. More precisely, we have the following.

Theorem 7.3. *Let $L > 0$ be given. Then there holds*

$$4\pi A \leq L^2 \tag{7.5}$$

for all simply closed planar curves $\mathcal{C}$, where equality holds in the "isoperimetric inequality" (7.5) if and only if $\mathcal{C}$ is a circle with circumference L.

Proving this result is fairly elementary, and we sketch the proof given in [26]. Before that, we shall need Wirtinger's inequality, which we state here without proof (but a proof follows from basic arguments using a Fourier series expansion).

Lemma 7.3. *If $y = y(t)$ is a 2π-periodic smooth function such that $\int_0^{2\pi} y(t)\,dt = 0$, then*

$$\int_0^{2\pi} y(t)\,dt \leq \int_0^{2\pi} \left(\frac{dy}{dt}\right)^2 dt,$$

where equality holds if and only if y is a linear combination of $\cos(t)$ and $\sin(t)$.

Proof of Theorem 7.3. For a simply closed curve $\mathcal{C}$, let $(x(t), y(t))$ for $a \leq t \leq b$ be a parameterization of this curve. Then the length of $\mathcal{C}$ is given by

$$L = \int_a^b ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt, \tag{7.6}$$

and the area of the region enclosed by $\mathcal{C}$ is given by the line integral

$$A = -\int_C y\,dx = -\int_a^b y\frac{dx}{dt}\,dt, \tag{7.7}$$

where we choose the positive orientation of $\mathcal{C}$ with respect to its enclosed interior.

Set $t = (2\pi/L)s$, and by a change of suitable coordinates, we may assume $\overline{y} := \int_0^{2\pi} y(t)\,dt = 0$. Then

$$\int_0^{2\pi} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 dt = \int_0^{2\pi} \left(\frac{ds}{dt}\right)^2 dt = \frac{L^2}{2\pi}.$$

Thus,

$$\begin{aligned} L^2 - 4\pi A &= 2\pi \int_0^{2\pi} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + 2y\frac{dx}{dt}\,dt \\ &= 2\pi \int_0^{2\pi} \left(\frac{dx}{dt} + y\right)^2 dt + 2\pi \int_0^{2\pi} \left(\frac{dy}{dt}\right)^2 - y^2\,dt \end{aligned}$$

Combining this with Lemma 7.3 and the fact $\int_0^{2\pi} \left(\frac{dx}{dt} + y\right)^2 dt \geq 0$, we get $L^2 - 4\pi A \geq 0$. Furthermore, $L^2 - 4\pi A = 0$ if and only if $\mathcal{C}$ is a circle with circumference L. □

In the general higher-dimensional setting, the isoperimetric problem in $\mathbb{R}^n$ entails minimizing the surface area of all domains with a prescribed volume, or equivalently, maximizing the volume among all domains whose boundary surface has fixed $(n-1)$-dimensional area. Some proper assumptions are needed to formulate the problem precisely. We consider connected domains $\Omega \subset \mathbb{R}^n$ whose boundary surface $S = \partial\Omega$ is a compact, smooth surface (at least C^2 is required). Thus, S is a closed surface, i.e., it is compact without boundary.

A solution of the isoperimetric problem for $n \geq 3$ does not appear to have a simple solution like that of the two-dimensional case. Nonetheless, a solution is derived using the calculus of variations. If $U \subset \mathbb{R}^{n-1}$ is open and $u \in C^2(U)$, then the graph of u,

$$\{(x, u(x)) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x \in U\},$$

defines a surface in $\mathbb{R}^n$. We suppose the surface S is described by such a function $u : U \longrightarrow \mathbb{R}$. Therefore, S is a level set of the mapping $(x, u(x)) \mapsto x_n - u(x)$ and the downward pointing unit normal vector $\nu(x)$ at $(x, u(x)) \in S$ is defined by

$$\nu(x) = \frac{(Du(x), -1)}{\sqrt{1 + |Du(x)|^2}}$$

and $\nu(x)$ belongs to the lower hemisphere

$$S_-^{n-1} = \{(y, -\sqrt{1 - |y|^2}) \,:\, y \in \mathbb{R}^{n-1},\ |y| < 1\}.$$

We treat $x = (x_1, x_2, \ldots, x_{n-1})$ and $y = (y_1, y_2, \ldots, y_{n-1})$ as the coordinates for S and S_-^{n-1}, respectively, and in these coordinates, the Gauss map is given by

$$x \mapsto y = \frac{Du(x)}{\sqrt{1 + |Du(x)|^2}}.$$

The second fundamental form of S is the Jacobian of the Gauss map

$$II(x) = \Big(\frac{\partial y_i}{\partial x_j}\Big)_{i,j}.$$

The principle curvatures of the surface S at x are defined to be the eigenvalues of the matrix $II(x)$. The mean curvature of S at x, denoted by $H(x)$, is the arithmetic average of the principle curvatures, i.e.,

$$H(x) = \frac{1}{n} Trace(II(x)) = \frac{1}{n} div\Big(\frac{Du(x)}{\sqrt{1 + |Du(x)|^2}}\Big).$$

7.4 Minimal Surfaces and Surfaces with constant mean curvature

7.5 Sharp Hardy-Littlewood-Sobolev inequalities in $\mathbb{R}^n$

As encountered numerous times already, knowing the precise value of the sharp constant to certain functional inequalities and embeddings can help study differential and integral equations and resolve fundamental problems in other branches of mathematics. In this section, we shall continue this focus and revisit a class of integral inequalities, the so-called Hardy-Littlewood-Sobolev (HLS) inequalities. Our objective is to provide accurate estimates on their best constants and explicitly calculate these constants when possible.

Recall in Section 1.3.2 of Chapter 1, we already introduced the HLS inequality and used Hardy-Littlewood maximal operators and the Marcinkiewicz interpolation inequalities to obtain the non-sharp version. Here, we use a different ideas to derive the HLS inequality and provide an upper bound on the sharp constant; and in the diagonal case, we explicitly compute the sharp constant. Our presentation follows that of Lieb and Loss [21] (also see [20]). Our strategy to finding the best constant uses a variational approach, which requires we classify the resulting extremal functions. Interestingly, the extremal functions are solutions to the Euler-Lagrange system (1.37) of integral equations.

Remark 7.4. *In general, it turns out all 'regular' solutions of* (1.37) *are essentially unique [6] and can be classified as having a very specific bubble form. This fact is a direct extension of the classification result for the critical Sobolev inequality and Lane-Emden equation.*

For convenience, we shall state a refined version of the HLS inequality.

Theorem 7.4 (Sharp HLS inequality). *Let $p, r > 1$ and $0 < \lambda < n$ such that $\frac{1}{p} + \frac{1}{r} + \frac{\lambda}{n} = 2$. Then there exists a sharp constant $C = C(n, \lambda, p)$ such that*

$$\Big| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda} h(y)\, dx\, dy \Big| \leq C \|f\|_{L^p(\mathbb{R}^n)} \|h\|_{L^p(\mathbb{R}^n)} \tag{7.8}$$

for every $f \in L^p(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. The sharp constant satisfies

$$C(n, \lambda, p) \leq \frac{n}{n-\lambda} \Big(\frac{|\mathbb{S}^{n-1}|}{n} \Big)^{\lambda/n} \frac{1}{pr} \Big(\Big(\frac{\lambda/n}{1-1/p} \Big)^{\lambda/n} + \Big(\frac{\lambda/n}{1-1/r} \Big)^{\lambda/n} \Big).$$

Moreover, in the diagonal case, i.e., if $p = r = 2n/(2n-\lambda)$, then

$$C(n, \lambda, p) = C(n, \lambda) = \pi^{\lambda/2} \frac{\Gamma(n/2 - \lambda/2)}{\Gamma(n - \lambda/2)} \Big\{ \frac{\Gamma(n/2)}{\Gamma(n)} \Big\}^{-1+\lambda/n}. \tag{7.9}$$

In this diagonal case, equality in (7.8) holds if and only if $h \equiv cf$, for some constant c, and

$$f(x) = A(\gamma^2 + |x-a|^2)^{-(2n-\lambda)/2}$$

for some $A \in \mathbb{C}$, $\gamma \in \mathbb{R} \backslash \{0\}$ and $a \in \mathbb{R}^n$.

7.5.1 Rough HLS inequality: a second proof

Some preliminary results are in order.

Proposition 7.1 (Layer Cake Representation). *Let ν be a (positive) measure on the Borel sets of the set $[0, \infty)$ such that*

$$\phi(t) = \nu([0, t))$$

is finite for each $t > 0$. Here, note $\phi(0) = 0$ and ϕ is Borel measurable since it is monotone increasing. Now, let $(X, \mathbf{M}, \mu)$ be a σ-finite measure space and f any non-negative measurable function on X. Then

$$\int_X \phi(f(x))\, d\mu = \int_0^\infty \mu(\{x \,|\, f(x) > t\})\, \nu(dt).$$

In particular, if $\nu = pt^{p-1}\, dt$ for $p > 0$, we have

$$\int_X f(x)^p\, d\mu = p \int_0^\infty t^{p-1} \mu(\{x \,|\, f(x) > t\})\, dt.$$

And by choosing μ to be the Dirac measure at some point $x \in \mathbb{R}^n$ and $p = 1$, there holds

$$f(x) = \int_0^\infty \chi_{\{f > t\}}(x)\, dt.$$

Proof. Define the level sets

$$S_f(t) := \{x \,:\, f(x) > t\}$$

and note that

$$\int_0^\infty \mu(S_f(t))\,\nu(dt) = \int_0^\infty \int_X \chi_{\{f>t\}}(x)\,\mu(dx)\nu(dt),$$

since the characteristic function $\chi_{\{f>t\}}(x)$ is jointly measurable. By Fubini's theorem and basic calculations, we derive

$$\begin{aligned}\int_0^\infty \int_X \chi_{\{f>t\}}(x)\,\mu(dx)\nu(dt) &= \int_X \Big(\int_0^\infty \chi_{\{f>t\}}(x)\,\nu(dt)\Big)\,\mu(dx)\\ &= \int_X \Big(\int_0^{f(x)} \nu(dt)\Big)\,\mu(dx) = \int_X [\phi(f(x))]\,\mu(dx)\end{aligned}$$

and this completes the proof. □

Proof of the Rough HLS inequality. To establish the non-optimal inequality (7.8), we may assume that $\|f\|_{L^p(\mathbb{R}^n)} = \|h\|_{L^r(\mathbb{R}^n)} = 1$. Thanks to the Layer Cake representation (Proposition 7.1), we may write

$$\begin{aligned}|x|^{-\lambda} &= \lambda \int_0^\infty c^{-(\lambda+1)}\chi_{\{|x|<c\}}(x)\,dc,\\ f(x) &= \int_0^\infty \chi_{\{f>a\}}(x)\,da,\\ h(x) &= \int_0^\infty \chi_{\{h>b\}}(x)\,db.\end{aligned}$$

Inserting these representations into the left-hand side of (7.8), we get

$$\begin{aligned}I &:= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda}h(y)\,dx\,dy\\ &= \lambda \int_0^\infty\int_0^\infty\int_0^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} c^{-(\lambda+1)}\chi_{\{f>a\}}(x)\chi_{\{h>b\}}(y)\chi_{\{|x|<c\}}(x-y)\,dx\,dy\,da\,db\,dc. \quad (7.10)\end{aligned}$$

Define

$$I(a,b,c) := \frac{v(a)w(b)u(c)}{\max\{v(a), w(b), u(c)\}},$$

where $v(a) = \int_{\mathbb{R}^n}\chi_{\{f>a\}}\,dx$, $w(b) = \int_{\mathbb{R}^n}\chi_{\{h>b\}}\,dx$ and $u(c) = \frac{|\mathbb{S}^{n-1}|}{n}c^n$. Noting that we may bound one of the characteristic functions in the integrand of (7.10), we deduce that

$$I \le \lambda \int_0^\infty\int_0^\infty\int_0^\infty c^{-(\lambda+1)} I(a,b,c)\,da\,db\,dc. \quad (7.11)$$

We may also rewrite the norms of f and g as

$$\|f\|^p_{L^p(\mathbb{R}^n)} = p\int_0^\infty a^{p-1}v(a)\,da = 1 \quad\text{and}\quad \|h\|^r_{L^r(\mathbb{R}^n)} = r\int_0^\infty b^{r-1}w(b)\,db = 1.$$

Now, if we assume $v(a) \geq w(b)$, we get the following estimate

$$\begin{aligned}
\int_0^\infty c^{-(\lambda+1)} I(a,b,c)\,dc &\leq \int_{u(c)\leq v(a)} c^{-(\lambda+1)} w(b)u(c)\,dc + \int_{u(c)>v(a)} c^{-(\lambda+1)} w(b)u(c)\,dc \\
&\leq w(b)\frac{|\mathbb{S}^{n-1}|}{n} \int_0^{(v(a)n/|\mathbb{S}^{n-1}|)^{1/n}} c^{-(\lambda+1)+n}\,dc + w(b)v(a) \int_{(v(a)n/|\mathbb{S}^{n-1}|)^{1/n}}^\infty c^{-(\lambda+1)}\,dc \\
&= \frac{1}{n-\lambda}(|\mathbb{S}^{n-1}|/n)^{\lambda/n} w(b)v(a)^{1-\lambda/n} + \frac{1}{\lambda}(|\mathbb{S}^{n-1}|/n)^{\lambda/n} w(b)v(a)^{1-\lambda/n} \\
&= \frac{n}{\lambda(n-\lambda)}(|\mathbb{S}^{n-1}|/n)^{\lambda/n} w(b)v(a)^{1-\lambda/n}.
\end{aligned}$$

We apply similar arguments for the case $w(b) \geq v(a)$ and collecting terms accordingly to get

$$I \leq \frac{n}{n-\lambda}(|\mathbb{S}^{n-1}|/n)^{\lambda/n} \int_0^\infty \int_0^\infty z(a,b)\,da\,db, \tag{7.12}$$

where

$$z(a,b) := \min\{w(b)v(a)^{1-\lambda/n},\, w(b)^{1-\lambda/n}v(a)\},$$

and we note that $w(b) \leq v(a)$ if and only if $w(b)v(a)^{1-\lambda/n} \leq w(b)^{1-\lambda/n}v(a)$. Splitting the integral on the right-hand side of (7.12) and applying Fubini's theorem, we get

$$\begin{aligned}
\int_0^\infty \int_0^\infty z(a,b)\,da\,db &\leq \int_0^\infty v(a) \int_0^{a^{p/r}} w(b)^{1-\lambda/n}\,db\,da + \int_0^\infty v(a)^{1-\lambda/n} \int_{a^{p/r}}^\infty w(b)\,db\,da \\
&= \int_0^\infty v(a) \int_0^{a^{p/r}} w(b)^{1-\lambda/n}\,db\,da + \int_0^\infty w(s) \int_0^{s^{r/p}} v(t)^{1-\lambda/n}\,dt\,ds \\
&=: I_1 + I_2.
\end{aligned}$$

We estimate the first integral term, I_1. Indeed, setting $m = (r-1)/(1-\lambda/n)$ and using the conjugate pair n/λ and $\frac{n}{n-\lambda}$, Hölder's inequality leads to

$$\int_0^{a^{p/r}} w(b)^{1-\lambda/n} b^m b^{-m}\,db \leq \Big(\int_0^{a^{p/r}} w(b)^{\frac{1-\frac{\lambda}{n}}{1-\frac{\lambda}{n}}}\,db\Big)^{1-\frac{\lambda}{n}} \Big(\int_0^\infty b^{-mn/\lambda}\,db\Big)^{\frac{\lambda}{n}}$$

and since $mn < \lambda$, there holds $\int_0^\infty b^{-mn/\lambda}\,db = \frac{\lambda}{\lambda - mn} = \frac{\lambda}{n - r(n-\lambda)} < \infty$. We then get

$$\begin{aligned}
I_1 &\le \int_0^\infty v(a)\Big(\big(\int_0^{a^{p/r}} w(b)\,db\big)^{1-\frac{\lambda}{n}}\big(\int_0^\infty b^{-mn/\lambda}\,db\big)^{\frac{\lambda}{n}}\Big)\,da\\
&\le \Big(\frac{\lambda}{n - r(n-\lambda)}\Big)^{\lambda/n}\Big(\int_0^\infty v(a)a^{p-1}\,da\Big)\Big(\int_0^\infty w(b)b^{r-1}\,db\Big)^{1-\lambda/n}\\
&= \Big(\frac{\lambda}{n - r(n-\lambda)}\Big)^{\lambda/n}\frac{\|f\|^p_{L^p(\mathbb{R}^n)}}{p}\Big(\frac{\|h\|^r_{L^r(\mathbb{R}^n)}}{r}\Big)^{1-\lambda/n}\\
&= \frac{1}{pr}\Big(\frac{\lambda r}{n - r(n-\lambda)}\Big)^{\lambda/n}\|f\|^p_{L^p(\mathbb{R}^n)}\|h\|^{r(1-\lambda/n)}_{L^r(\mathbb{R}^n)} = \frac{1}{pr}\Big(\frac{\lambda/n}{1/r - 1 + \lambda/n)}\Big)^{\lambda/n}\\
&= \frac{1}{pr}\Big(\frac{\lambda/n}{1 - 1/p}\Big)^{\lambda/n}.
\end{aligned}$$

Analogous symmetric arguments will also lead to $I_2 \le \frac{1}{pr}\Big(\frac{\lambda/n}{1-1/r}\Big)^{\lambda/n}$. By combining these estimates for I_1 and I_2, we deduce

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda}h(y)\,dx\,dy\Big| &\le \frac{n}{n-\lambda}\Big(\frac{|\mathbb{S}^{n-1}|}{n}\Big)^{\lambda/n}(I_1 + I_2)\\
&\le \frac{n}{n-\lambda}\Big(\frac{|\mathbb{S}^{n-1}|}{n}\Big)^{\lambda/n}\frac{1}{pr}\Big(\Big(\frac{\lambda/n}{1-1/p}\Big)^{\lambda/n} + \Big(\frac{\lambda/n}{1-1/r}\Big)^{\lambda/n}\Big).
\end{aligned}$$

This establishes the rough HLS inequality. □

7.5.2 Conformal invariance of the diagonal HLS inequality

We now focus on the diagonal case $p = r = 2n/(n-\lambda)$ in the HLS inequality (7.8), in which we may exploit conformal invariance and symmetry properties to explicitly calculate the best constant (7.9) and classify the functions f and g that achieve it. For the reader's convenience, we go over some necessary prerequisite material.

Rearrangements and rearrangement inequalities

We introduce the notion of a symmetric rearrangement of a given set or a function. The proper class of functions for introducing our notion of rearrangement are the Borel measurable functions that vanish at infinity. Namely, if $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a Borel measurable function, then f is said to vanish at infinity if the (n-dimensional) Lebesgue measure $|\{x \in \mathbb{R}^n : |f(x)| > t\}|$ is finite for each $t > 0$.

If $A \subset \mathbb{R}^n$ is a Borel set and $|A| < \infty$, we define A^*, the symmetric rearrangement of the set A, to be the open ball $B_r(0)$ whose volume is equal to $|A|$, i.e.,

$$A^* = \{x \in \mathbb{R}^n : |x| < r\}$$

such that

$$|A| = |B_r(0)| = \frac{|\mathbb{S}^{n-1}|}{n} r^n.$$

The symmetric-decreasing rearrangement of a characteristic function of a set A is given by

$$\chi_A^* = \chi_{A^*},$$

which can be shown to be lower semicontinuous. Notice that if $f : X \longrightarrow \mathbb{R}$ is a measurable function on a measurable space $(X, \mathcal{M})$, then the level set

$$S_f(t) = \{x \in X \,:\, f(x) > t\} \quad \text{for each } t \in \mathbb{R}$$

is measurable, i.e. $S_f(t) \in \mathcal{M}$. Then recall f is lower semicontinuous if $S_f(t)$ is open for each $t \in \mathbb{R}$.

If $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a Borel measurable function vanishing at infinity, and thanks to the Layer Cake Representation Theorem, we define f^*, the symmetric-decreasing rearrangement of f, by

$$f^*(x) = \int_0^\infty \chi^*_{\{|f|>t\}}(x)\, dt. \tag{7.13}$$

It is interesting to compare this with the fact that

$$|f(x)| = \int_0^\infty \chi_{\{|f|>t\}}(x)\, dt.$$

The term given to the rearrangement f^* can be explained by the following useful properties.

Proposition 7.2. *Given a Borel measurable function f that vanishes at infinity, its symmetric-decreasing rearrangement f^* is a non-negative lower semicontinuous (and therefore a measurable) function. Moreover, f^* is radially symmetric and non-increasing.*

Proof. It is obvious f^* is non-negative from the definition (7.13). For each $t \in \mathbb{R}$, we show $E_t := S_{f^*}(t) = \{z \,:\, f^*(z) > t\}$ is open and therefore a Borel set, namely, it suffices to show $E_t^c = \{z \in \mathbb{R}^n \,:\, f^*(z) \leq t\}$ is closed. Pick $t > 0$ and suppose the sequence $\{x_n\}_{n=1}^\infty \subset E_t^c$ converges to some point x. Then

$$t \geq f^*(x_n) = \int_0^\infty \chi^*_{\{|f|>s\}}(x_n)\, ds = \int_0^\infty \chi_{B_{r_s}(0)}(x_n)\, ds.$$

By Fatou's lemma,

$$t \geq \liminf_{n\to\infty} \int_0^\infty \chi_{B_{r_s}(0)}(x_n)\, ds \geq \int_0^\infty \chi_{B_{r_s}(0)}(x)\, ds,$$

and this verifies $x \in E_t^c$ and therefore E_t^c is a closed subset. This also proves the lower semicontinuity property of f^*.

Assume $|x| = |y|$ and let r_t so that $|\{z : |f(z)| > t\}| = |B_{r_t}(0)|$.

$$f^*(x) = \int_0^\infty \chi^*_{\{|f|>t\}}(x)\,dt = \int_0^\infty \chi_{B_{r_t}(0)}(x)\,dt = \int_0^\infty \chi_{B_{r_t}(0)}(x)\,dt = \int_0^\infty \chi^*_{\{|f|>t\}}(y)\,dt = f^*(y)$$

and this verifies f^* is radially symmetric. Likewise, if $|x| \le |y|$, then $\chi_{B_{r_t}(0)}(x) \ge \chi_{B_{r_t}(0)}(y)$. Therefore,

$$f^*(x) = \int_0^\infty \chi^*_{\{|f|>t\}}(x)\,dt = \int_0^\infty \chi_{B_{r_t}(0)}(x)\,dt \ge \int_0^\infty \chi_{B_{r_t}(0)}(y)\,dt = \int_0^\infty \chi^*_{\{|f|>t\}}(y)\,dt = f^*(y),$$

that is, f^* is non-increasing. □

Proposition 7.3. *The level sets of f^* are just rearrangements of the level sets of f^*, i.e.,*

$$\{x \in \mathbb{R}^n : f^*(x) > t\} = \{x \in \mathbb{R}^n : |f(x)| > t\}.$$

Proposition 7.4. *For any measurable function $\phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are monotone non-decreasing, $\phi_1(0) = \phi_2(0) = 0$ and either $\int_{\mathbb{R}^n} \phi_1(|f(x)|)\,dx$ or $\int_{\mathbb{R}^n} \phi_2(|f(x)|)\,dx$ is finite, there holds*

$$\int_{\mathbb{R}^n} \phi(|f(x)|)\,dx = \int_{\mathbb{R}^n} \phi(|f^*(x)|)\,dx. \tag{7.14}$$

Consequently, for $f \in L^p(\mathbb{R}^n)$

$$\|f\|_{L^p(\mathbb{R}^n)} = \|f^*\|_{L^p(\mathbb{R}^n)} \quad \text{for all } 1 \le p \le \infty.$$

Proposition 7.5. *Let f and g be non-negative measurable functions on $\mathbb{R}^n$ vanishing at infinity, and suppose $f(x) \le g(x)$ for all $x \in \mathbb{R}^n$. Then*

$$f^*(x) \le g^*(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{7.15}$$

Theorem 7.5 (Nonexpansivity of rearrangements)**.** *Let $J : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-negative convex function such that $J(0) = 0$, and let f and g be non-negative measurable functions on $\mathbb{R}^n$ that vanish at infinity. Then*

$$\int_{\mathbb{R}^n} J(f^*(x) - g^*(x))\,dx \le \int_{\mathbb{R}^n} J(f(x) - g(x))\,dx. \tag{7.16}$$

Particularly, if we also assume J is strictly convex and $f = f^*$, and f is strictly decreasing, then equality in (7.16) *implies that $g = g^*$.*

Isometries of the unit sphere and the conformal group

The sharp HLS inequality enjoys rich symmetry and invariant properties essential in its proof. We describe such properties carefully.

Given a function $f(x)$ defined in $\mathbb{R}^n$, and a given point $a \in \mathbb{R}^n$, we define the translation operator by $\tau_a f(x) = f(x-a)$. If $\mathcal{R} \in O(n)$, where $O(n)$ is the orthogonal group of rotations and reflections of $\mathbb{R}^n$, we define the operator by $\mathcal{R}f(x) = f(\mathcal{R}^{-1}x)$. A simple calculation will reveal that inequality (7.8) is invariant under translations, rotations and reflections. By this we mean, for instance, if we replace $f(x)$ and $h(x)$ by $\tau_a f(x)$ and $\mathcal{R}f(x)$ and $\mathcal{R}h(x)$, respectively, in (7.8), we see that both sides of the HLS inequality does not change. A similar property holds for reflections and rotations. More generally, the HLS inequality is invariant under the following action of the Euclidean group:

$$[(\mathcal{R}, a), f(x)] \mapsto f(\mathcal{R}^{-1}x - a) \text{ for } \mathcal{R} \in O(n),\, a \in \mathbb{R}^n,$$

and similarly for h. The Euclidean group of transformations define rigid motion, which means geometric figures in Euclidean space remain congruent under such actions.

We define the dilation or scaling operator $f_\delta(x) = \delta^\theta f(\delta x)$ for $\delta > 0$, where θ is some fixed positive number. We consider the specific scalings $f_\delta(x) := \delta^{n/p} f(\delta x)$ and $h_\delta(x) := \delta^{n/r} h(\delta x)$. Then the Lebesgue norms are preserved by these scalings, i.e.,

$$\|f_\delta\|_{L^p(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |\delta^{n/p} f(\delta x)|^p\, dx\Big)^{1/p} = \delta^{n/p}\Big(\int_{\mathbb{R}^n} |f(y)|^p \delta^{-n}\, dy\Big)^{1/p} = \|f\|_{L^p(\mathbb{R}^n)},$$

where we used the change of variables $y = \delta x$. Likewise, we can show $\|h_\delta\|_{L^p(\mathbb{R}^n)} = \|h\|_{L^p(\mathbb{R}^n)}$. The left-hand side of the HLS inequality is also invariant with respect to these scalings, since

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \delta^{n/p} f(\delta x)|x-y|^{-\lambda}\delta^{r/n} h(\delta x)\, dxdy\Big| &= \delta^{\frac{n}{p}+\frac{n}{r}+\lambda}\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\delta x)|\delta x - \delta y|^{-\lambda} h(\delta x)\, dxdy\Big| \\ &= \delta^{\frac{n}{p}+\frac{n}{r}+\lambda}\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\bar{x})|\bar{x}-\bar{y}|^{-\lambda} h(\bar{x})\lambda^{-2n}\, d\bar{x}d\bar{y}\Big| \\ &= \Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda} h(x)\, dxdy\Big|,\end{aligned}$$

where we used the change of variables $\bar{x} = \delta x$ and $\bar{y} = \delta y$ and the fact that $\frac{n}{p} + \frac{n}{r} + \lambda = 2n$. We have verified that (7.8) has scaling symmetry in the sense if we replace $f(x)$ and $h(x)$ by $f_\delta(x)$ and $h_\delta(x)$, then the inequality persists.

The reader may wonder if the Euclidean group defines all the invariants for the HLS inequality. Recall that stretching is a

7.5.3 Proof of the sharp HLS inequality

APPENDIX A

Basic Inequalities, Sobolev Embeddings, and Convergence Theorems

This appendix covers some basic inequalities, embeddings and convergence results that we frequently apply throughout.

A.1 Basic Inequalities

Theorem A.1 (Cauchy's inequality)**.** *There holds for $a, b \in \mathbb{R}$,*

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}.$$

More generally, we have Cauchy's inequality with ϵ:

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon} \quad (a, b > 0,\ \epsilon > 0).$$

We shall need to recall the definition of a convex real-valued function.

Definition A.1. *We say a set U in $\mathbb{R}^n$ is convex if $\tau x + (1-\tau)y \in U$ for all $x, y \in U$ and all $0 \leq \tau \leq 1$. For a convex set U, we say a real-valued function f is convex on U if*

$$f(\tau x + (1-\tau)y) \leq \tau f(x) + (1-\tau) f(y) \quad \text{for all } x, y \in U \quad \text{and all } \tau \in (0,1).$$

If the above inequality is strict, then we say f is a strictly convex function on U.

Theorem A.2 (Young's inequality)**.** *Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

and equality holds above if and only if $a^p = b^q$.

Proof. Take $p, q > 1$ so that $1/p + 1/q = 1$, and assume a and b is a non-trivial pair of positive reals, since the inequality is trivial otherwise. Set $\tau = 1/p$ and $1 - \tau = 1/q$. Since the logarithmic function $f(x) = -\log(x)$ is convex on $(0, \infty)$, we get

$$\log(ab) = \log(a) + \log(b) = \tau \log(a^p) + (1-\tau)\log(b^q) \leq \log(\tau a^p + (1-\tau)b^q), \qquad \text{(A.1)}$$

with equality if and only if $a^p = b^q$. The desired result follows after applying the exponential function to (A.1). □

Theorem A.3 (Hölder's inequality)**.** *Assume* $1 \leq p, q \leq \infty$ *and* $1/p + 1/q = 1$. *If* $u \in L^p(U)$, $v \in L^q(U)$, *then*

$$\int_U |uv|\, dx \leq \|u\|_{L^p(U)} \|v\|_{L^q(U)}. \qquad \text{(A.2)}$$

Equality holds if and only if $\alpha|f|^p = \beta|g|^q$ *almost everywhere for some constants* α, β *with* $(\alpha, \beta) \neq (0, 0)$.

Proof. Let $u \in L^p(U)$ and $v \in L^q(U)$. From the homogeneity of the L^p norms, we can assume that $\|u\|_{L^p(U)} = \|v\|_{L^q(U)} = 1$. Then by Young's inequality of Theorem A.2,

$$\int_U |uv|\, dx \leq \frac{1}{p}\int_U |u|^p\, dx + \frac{1}{q}\int_U |v|^q\, dx = \frac{1}{p} + \frac{1}{q} = 1 = \|u\|_{L^p(U)}\|v\|_{L^q(U)}.$$

□

An easy extension of this inequality is the following whose proof we omit.

Theorem A.4 (General Hölder's Inequality)**.** *Let* $1 \leq p_1, p_2, \ldots, p_m \leq \infty$ *with* $\sum_{k=1}^m \frac{1}{p_k} = 1$, *and assume* $u_k \in L^{p_k}(U)$ *for* $k = 1, \ldots, m$. *Then*

$$\int_U |u_1 \ldots u_m|\, dx \leq \prod_{k=1}^m \|u_k\|_{L^{p_k}(U)}. \qquad \text{(A.3)}$$

Theorem A.5 (Jensen's inequality)**.** *Assume* $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ *is convex in* $\mathbb{R}^m$, *and* $U \subset \mathbb{R}^n$ *is bounded and open. Let* $\mathbf{u} : U \longrightarrow \mathbb{R}^m$ *be summable. Then*

$$f\Big(\frac{1}{|U|}\int_U \mathbf{u}\, dx\Big) \leq \frac{1}{|U|}\int_U f(\mathbf{u})\, dx.$$

Theorem A.6 (L^p interpolation)**.** *Assume that* $1 \leq p \leq r \leq q \leq \infty$ *and*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{(1-\theta)}{q}.$$

Suppose also that $u \in L^p(U) \cap L^q(U)$. *Then* $u \in L^r(U)$ *and*

$$\|u\|_{L^r(U)} \leq \|u\|_{L^p(U)}^{\theta} \|u\|_{L^q(U)}^{1-\theta}. \qquad \text{(A.4)}$$

Proof. Since $\frac{\theta r}{p} + \frac{(1-\theta)r}{q} = 1$, Hölder's inequality yields

$$\int_U |u|^r\,dx = \int_U |u|^{\theta r}|u|^{(1-\theta)r}\,dx \le \left(\int_U |u|^{\theta r \frac{p}{\theta r}}\right)^{\frac{\theta r}{p}} \left(\int_U |u|^{(1-\theta)r\frac{q}{(1-\theta)r}}\,dx\right)^{\frac{(1-\theta)r}{q}}.$$

□

Theorem A.7 (Gronwall's inequality)**.** *Let $\eta(\cdot)$ be a non-negative absolutely continuous (i.e., differentiable a.e.) function on $[0,T]$, which satisfies for a.e. t, the differential inequality*

$$\eta'(t) \le \phi(t)\eta(t) + \psi(t), \tag{A.5}$$

where $\phi(t)$ and $\psi(t)$ are non-negative, summable functions on $[0,T]$. Then

$$\eta(t) \le e^{\int_0^t \phi(s)\,ds}\Big(\eta(0) + \int_0^t \psi(s)\,ds\Big) \text{ for all } 0 \le t < T. \tag{A.6}$$

In particular, if $\eta(0) = 0$ and

$$\eta'(t) \le \phi(t)\eta(t) \text{ for a.e. } t \in [0,T],$$

then

$$\eta \equiv 0 \text{ on } [0,T].$$

Proof. From (A.5),

$$\frac{d}{ds}\Big(\eta(s)e^{-\int_0^s \phi(r)\,dr}\Big) = e^{-\int_0^s \phi(r)\,dr}(\eta'(s) - \phi(s)\eta(s)) \le e^{-\int_0^s \phi(r)\,dr}\psi(s) \text{ for a.e. } 0 \le s \le T.$$

Integrating this we get, for each $0 \le t \le T$,

$$\eta(t)e^{-\int_0^t \phi(r)\,dr} \le \eta(0) + \int_0^t e^{-\int_0^s \phi(r)\,dr}\psi(s)\,ds \le \eta(0) + \int_0^t \psi(s)\,ds.$$

□

Sometimes, it is more convenient to use the integral form of Gronwall's inequality.

Theorem A.8. *Let $\xi(t)$ be a non-negative, summable function on $[0,T]$ which satisfies, for a.e. t the integral inequality*

$$\xi(t) \le C_1 \int_0^t \xi(s)\,ds + C2, \tag{A.7}$$

for some constants $C_1, C_2 \ge 0$. Then

$$\xi(t) \le C_2(1 + C_1 t e^{C_1 t}) \text{ for a.e. } 0 \le t \le T.$$

In particular, if

$$\xi(t) \le C1 \int_0^t \xi(s)\,ds \text{ for a.e. } 0 \le t \le T,$$

then

$$\xi \equiv 0 \text{ on } [0,T].$$

Proof. Set $\eta(t) = \int_0^t \xi(s)\, ds$ so that $\eta'(t) \le C_1\eta(t) + C_2$ for a.e. t in $[0,T]$. According to the differential version of Gronwall's inequality, we obtain

$$\eta(t) \le e^{C_1 t}(\eta(0) + C_2 t) = C_2 t e^{C_1 t}.$$

The result then follows from (A.7) since

$$\xi(t) \le C_1\eta(t) + C_2 \le C_2(1 + C_1 t e^{C_1 t}).$$

□

A.2 Sobolev Spaces and Sobolev Inequalities

We define the notion of weak derivatives then use it to define the so-called Sobolev spaces. Suppose $u, v \in L^1_{loc}(U)$ and $\alpha \in \mathbb{N}^n_+$ is a multi-index. We say that v is the αth-weak partial derivative of u, written $D^\alpha u = v$, if

$$\int_U u D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_U v\phi\, dx \quad \text{for all } \phi \in C_c^\infty(U).$$

Let $k \in \mathbb{N} \cup \{0\}$ and $1 \le p \le \infty$.

Definition A.2. *The Sobolev space $W^{k,p}(U)$ consists of all equivalence classes of locally summable functions $u : U \longrightarrow \mathbb{R}$ such that for each multi-index α with $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and belongs to the Lebesgue space $L^p(U)$. If $p = 2$, we will write $H^k(U) = W^{k,2}(U)$. For $1 \le p < \infty$, we equip $W^{k,p}(U)$ with the norm*

$$\|u\|_{W^{k,p}(U)} = \Big(\sum_{|\alpha| \le k} \int_U |D^\alpha u|^p\, dx \Big)^{1/p} \quad \textit{for } u \in W^{k,p}(U).$$

For $p = \infty$ we define the norm of $W^{k,\infty}$ by

$$\|u\|_{W^{k,\infty}(U)} = \sum_{|\alpha| \le k} \operatorname*{ess\,sup}_U |D^\alpha u| \quad \textit{for } u \in W^{k,p}(U).$$

Theorem A.9. *For each $k \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space. In particular, $H^k(U)$ equipped with the inner product*

$$\langle u, v \rangle_{H^k(U)} = \sum_{|\alpha| \le k} \int_U D^\alpha u \cdot D^\alpha v\, dx \quad \textit{for } u, v \in H^k(U),$$

is a Hilbert space.

Definition A.3. *We define $W_0^{k,p}(U)$ to be the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$. If $p = 2$, we write $H_0^k(U) = W_0^{k,2}(U)$.*

Actually, there is a historical connection between the notation above. Namely, for some time, it was not clear the connection between the Sobolev space $W^{k,p}(U)$ and the space $H^{k,p}(U)$ defined as the completion of $C^k(U)$ with respect to the $\|\cdot\|_{W^{k,p}(U)}$ norm. In 1964, Myers and Serrin established that $W^{k,p}(U) \equiv H^{k,p}(U)$ for all domains $U \subseteq \mathbb{R}^n$. This is a consequence of the fact that $C^\infty(U) \cap W^{k,p}(U)$ is dense in $W^{k,p}(U)$ for $1 \leq p < \infty$ and thus $W^{k,p}(U)$ is separable. We revisit this below but focus mainly on the case when $k = 1$.

We can extend the H^k spaces to the fractional-order Sobolev spaces in $\mathbb{R}^n$ with the help of the Fourier transform. We define such fractional Sobolev spaces here but an alternative on more general construction can be done using the Riesz potentials, which also yields corresponding Sobolev inequalities (see Section 3.1.5).

Definition A.4. *Let $0 < s < \infty$. We define the Sobolev space $H^s(\mathbb{R}^n)$ to be the collection of functions $u \in L^2(\mathbb{R}^n)$ such that $(1 + |y|^s)\widehat{u} \in L^2(\mathbb{R}^n)$ and $\|u\|_{H^s(\mathbb{R}^n)} = \|(1 + |y|^s)\widehat{u}\|_{L^2(\mathbb{R}^n)}$, where $\widehat{u} = \mathcal{F}[u]$.*

Here, $\mathcal{F}[\cdot]$ is the Fourier transform

$$\mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx.$$

The following verifies that we recover the usual spaces when the order$s$ is a non-negative integer k.

Theorem A.10. *Let $k \in \mathbb{N} \cup \{0\}$. A function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if*

$$(1 + |y|^k)\widehat{u} \in L^2(\mathbb{R}^n).$$

Moreover, there exists a positive constant C such that

$$C^{-1}\|u\|_{H^k(\mathbb{R}^n)} \leq \|(1 + |y|^k)\widehat{u}\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)} \quad \text{for each } u \in H^k(\mathbb{R}^n).$$

Next, we derive a couple of essential inequalities we will need to establish the so-called Sobolev embeddings. Specifically, we introduce and prove the Gagliardo–Nirenberg–Sobolev inequality and Morrey's inequality. For each estimate, we establish its corresponding Sobolev embedding theorems.

Theorem A.11 (Gagliardo-Nirenberg-Sobolev)**.** *Assume $1 \leq p < n$ and denote $p^* := np/(n-p)$. There exists a constant $C = C(n,p)$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C(n,p)\|Du\|_{L^p(\mathbb{R}^n)} \tag{A.8}$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Remark A.1. *Note that the functions u must have compact support to discriminate from obvious cases such as constant functions. However, it is interesting that the constant C does not depend on the size of the support of u.*

Proof. **Step 1:** We first prove the estimate for $p = 1$.

Since u has compact support, for each $i = 1, 2, \ldots, n$ and $x \in \mathbb{R}^n$ we have

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)\, dy_i;$$

and so for $i = 1, 2, \ldots, n$,

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, \ldots, y_i, \ldots, x_n)|\, dy_i.$$

Therefore,

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \ldots, y_i, \ldots, x_n)|\, dy_i \right)^{\frac{1}{n-1}}.$$

Integrating this inequality with respect to x_1 yields

$$\begin{aligned} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1 &\leq \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\ &= \left(\int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\ &\leq \left(\int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1\, dy_i \right)^{\frac{1}{n-1}}, \end{aligned} \tag{A.9}$$

where we used the general Hölder's inequality in the last inequality. Now integrate (A.9) with respect to x_2:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1\, dx_2 \leq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1\, dx_2 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=1, i\neq 2}^{n} I_i^{\frac{1}{n-1}}\, dx_2,$$

where

$$I_1 = \int_{-\infty}^{\infty} |Du|\, dy_1, \quad I_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1\, dy_i \quad (i = 3, 4, \ldots, n).$$

Applying the general Hölder's inequality once more to this yields

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1\, dx_2 \leq & \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1\, dx_2 \right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dy_1\, dx_2 \right)^{\frac{1}{n-1}} \\ & \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1\, dx_2\, dy_i \right)^{\frac{1}{n-1}} \end{aligned}$$

We continue integrating with respect to $x_3, x_4, \ldots, x_n$, until we arrive at

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\, dx \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du|\, dx_1 \ldots dy_i, \ldots dx_n \right)^{\frac{1}{n-1}}$$

$$= \left(\int_{\mathbb{R}^n} |Du|\, dx \right)^{\frac{n}{n-1}}. \tag{A.10}$$

Hence, this proves the theorem for $p = 1$.

Step 2: Consider the case where $p \in (1, n)$. If we apply estimate (A.10) to $v := |u|^{\gamma}$ ($\gamma > 1$ is to be determined below), we obtain

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\, dx \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |Dv|\, dx = \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |Du|\, dx$$

$$\leq \gamma \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}}\, dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p\, dx \right)^{\frac{1}{p}}. \tag{A.11}$$

Set

$$\gamma = \frac{p(n-1)}{n-p} > 1$$

so that

$$\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} = \frac{np}{n-p} = p^*.$$

Thus, (A.11) becomes

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx \right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^n} |Du|^p\, dx \right)^{\frac{1}{p}}$$

and this completes the proof. □

Theorem A.12 (Morrey's inequality)**.** *Assume $n < p \leq \infty$. Then there exists a constant $C(n, p)$ such that*

$$\|u\|_{C^{0,1-n/p}(\mathbb{R}^n)} \leq C(n, p) \|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{A.12}$$

for all $u \in C^1(\mathbb{R}^n)$.

Proof. **Step 1:** We claim there exists a constant $C = C(n)$ depending only on n such that

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y) - u(x)|\, dy \leq C \int_{B_r(x)} \frac{|Du(y)|}{|y - x|^{n-1}}\, dy \tag{A.13}$$

for each open ball $B_r(x) \subset \mathbb{R}^n$.

To show this, fix any point $w \in \partial B_1(0)$. Then, if $0 < s < r$,

$$\begin{aligned} |u(x+sw) - u(x)| &= \left| \int_0^s \frac{d}{dt} u(x+tw)\, dt \right| = \left| \int_0^s Du(x+tw) \cdot w \, dt \right| \\ &\le \int_0^s |Du(x+tw)|\, dt. \end{aligned}$$

Hence,

$$\int_{\partial B_1(0)} |u(x+sw) - u(x)|\, ds_w \le \int_0^s \int_{\partial B_1(0)} |Du(x+tw)|\, ds_w\, dt. \tag{A.14}$$

We estimate the right-hand side of this inequality to get

$$\begin{aligned} \int_0^s \int_{\partial B_1(0)} |Du(x+tw)|\, ds_w\, dt &= \int_0^s \int_{\partial B_t(x)} \frac{|Du(y)|}{t^{n-1}}\, ds_y\, dt \\ &= \int_{B_s(x)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy \le \int_{B_r(x)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy, \end{aligned}$$

where $y = x + tw$ and $t = |x-y|$. The left-hand side can be written

$$\int_{\partial B_1(0)} |u(x+sw) - u(x)|\, ds_w = \frac{1}{s^{n-1}} \int_{\partial B_s(x)} |u(z) - u(x)|\, ds_z,$$

where $z = x + sw$. Combining the preceding two calculations in (A.14), we obtain the estimate

$$\int_{\partial B_s(x)} |u(z) - u(x)|\, ds_z \le s^{n-1} \int_{B_r(x)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy.$$

Integrate this with respect to s from 0 to r yields

$$\int_{B_r(x)} |u(y) - u(x)|\, dy \le \frac{r^n}{n} \int_{B_r(x)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy.$$

This proves our first claim.

Step 2: Fix $x \in \mathbb{R}^n$. Applying estimate (A.13) then Hölder's inequality, we get

$$\begin{aligned} |u(x)| &\le \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(x) - u(y)|\, dy + \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(y)|\, dy \\ &\le C \int_{B_1(x)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy + C\|u\|_{L^p(B_1(x))} \\ &\le C \left(\int_{\mathbb{R}^n} |Du|^p\, dy \right)^{\frac{1}{p}} \left(\int_{B_1(x)} \frac{1}{|x-y|^{(n-1)\frac{p}{p-1}}}\, dy \right)^{\frac{p-1}{p}} + C\|u\|_{L^p(\mathbb{R}^n)} \\ &\le C\|u\|_{W^{1,p}(\mathbb{R}^n)}. \end{aligned}$$

The last estimate holds since $p > n$ implies $(n-1)\frac{p}{p-1} < n$, so that

$$\int_{B_1(x)} \frac{1}{|x-y|^{(n-1)\frac{p}{p-1}}}\, dy < \infty.$$

As $x \in \mathbb{R}^n$ was chosen arbitrarily, there holds

$$\sup_{x\in\mathbb{R}^n} |u(x)| \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Step 3: Next, choose any two points $x, y \in \mathbb{R}^n$ and set $r := |x-y|$. Let $W := B_r(x) \cap B_r(y)$. Then

$$|u(x)-u(y)| \le \frac{1}{|W|}\int_W |u(x)-u(z)|\, dz + \frac{1}{|W|}\int_W |u(y)-u(z)|\, dz = I_1 + I_2.$$

Furthermore, estimate (A.13) allows us to estimate

$$\begin{aligned} I_1 = \frac{1}{|W|}\int_W |u(x)-u(z)|\, dz &\le C\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} |u(x)-u(z)|\, dz\right) \\ &\le C\left(\int_{B_r(x)} |Du|^p\, dz\right)^{\frac{1}{p}} \left(\int_{B_r(x)} \frac{dz}{|x-z|^{(n-1)\frac{p}{p-1}}}\right)^{\frac{p-1}{p}} \\ &\le C\left(r^{n-(n-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \|Du\|_{L^p(\mathbb{R}^n)} \\ &\le Cr^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Similarly, we calculate

$$I_2 = \frac{1}{|W|}\int_W |u(y)-u(z)|\, dz \le Cr^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)}$$

Hence,

$$|u(x)-u(y)| \le Cr^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)} = C|x-y|^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)},$$

therefore,

$$[u]_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} = \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}} \le C\|Du\|_{L^p(\mathbb{R}^n)}.$$

□

A.2.1 Extension and Trace Operators

Although we use the Gagliardo-Nirenberg-Sobolev and Morrey inequalities to prove the classical Sobolev embedding theorems, we shall also make use of the following basic results.

Theorem A.13 (Extension Theorem)**.** *Assume U is bounded and ∂U is C^1. Select a bounded open set V such that $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \longrightarrow W^{1,p}(\mathbb{R}^n)$$

such that for each $u \in W^{1,p}(U)$ there hold

(a) $Eu = u$ a.e. in U,

(b) Eu has support within V,

(c) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$

with the positive constant $C = C(p, U, V)$ depending only on p, U and V. Here Eu is called an extension of u to $\mathbb{R}^n$.

Theorem A.14 (Trace Theorem)**.** *Assume U is bounded and ∂U is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(U) \longrightarrow L^p(\partial U)$$

such that

(a) $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$,

(b) $\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$ for each $u \in W^{1,p}(U)$

with the positive constant $C = C(p, U)$ depending only on p and U.

Remark A.2. *The trace operator T enables us to assign boundary values along ∂U to functions in $W^{1,p}(U)$. This is especially useful for studying the Dirichlet problem and characterizing the space $W_0^{1,p}(U)$, the closure of $C_c^\infty(U)$ in $W^{1,p}(U)$, as the $W^{1,p}$ functions vanishing at the boundary. For example, if U is bounded and ∂U is C^1, and $u \in W^{1,p}(U)$, then (see [9][Theorem 2 on page 273])*

$$u \in W_0^{1,p}(U) \text{ if and only if } Tu = 0 \text{ on } \partial U.$$

Not surprisingly, the above extension and trace theorems can be extended to the higher-order cases, and we state these generalized results without proof but refer the reader to the reference [1]. Prior to doing so, some definitions and notation are needed. For any positive integer k, $1 < p < \infty$ and an open and bounded domain U with C^k boundary ∂U, we denote by $W^{k-1/p,p}(\partial\Omega)$ the space of "traces" $T(u) = u|_{\partial U}$ of functions u in $W^{k,p}(U)$, and we treat it as the collection of equivalence classes $\{[u] + W_0^{k,p}(U) \,|\, u \in W^{k,p}(U)\}$ equipped with the norm

$$\|T(u)\|_{W^{k-1/p,p}(\partial U)} := \inf_{u-v \in W_0^{k,p}(U)} \|v\|_{W^{k,p}(U)}.$$

Evidently, $W^{k-1/p,p}(\partial U)$ is a Banach space. As per the usual convention, we shall write $W^{k-1/2,2}(\partial U) = H^{k-1/2}(\partial U)$ whenever $p = 2$. Then the following extension result holds.

Theorem A.15. *Let $1 < p < \infty$, $k \geq 1$ and U is an open and bounded domain with C^k boundary ∂U. Then there exists a bounded linear operator $Ext : W^{k-1/p,p}(\partial U) \longrightarrow W^{k,p}(U)$ such that $Ext(u)|_{\partial U} = u$ for each $u \in W^{k-1/p,p}(\partial U)$.*

In the special case $k = 1$ and $p = 2$, the trace operator $T : H^1(U) \longrightarrow H^{1/2}(\partial U)$ with $u \longrightarrow u|_{\partial U}$ is a linear isometry from the closed orthogonal complement of $H_0^1(U)$ in $H^1(U)$ onto $H^{1/2}(\partial U)$. Thus, by the open mapping theorem, we can extend this to a bounded operator $Ext : H^{1/2}(\partial U) \longrightarrow H^1(U)$.

Furthermore, the following embeddings hold.

Theorem A.16. *Let $1 < p < \infty$, $k \geq 1$ and U is an open and bounded domain with C^k boundary ∂U. Then*

$$W^{k,p}(\partial U) \hookrightarrow W^{k-1/p,p}(\partial U) \hookrightarrow W^{k-1,p}(\partial U)$$

where the embeddings are compact. In particular, we have the compact embedding

$$H^1(U) \hookrightarrow L^2(\partial U).$$

A.2.2 Density of smooth functions in Sobolev spaces

The next property concerns the global approximation of functions in $W^{1,p}(U)$ by smooth functions.

Theorem A.17 (Density Theorem)**.** *Assume that U is bounded and suppose that $u \in W^{1,p}(U)$ for some $1 \leq p < \infty$.*

(a) There exists functions $u_m \in C^\infty(U) \cap W^{1,p}(U)$ such that

$$u_m \longrightarrow u \text{ in } W^{1,p}(U).$$

(b) If, in addition, ∂U is C^1, then statement (a) holds but the approximating sequence of functions can be taken to be smooth up to the boundary, i.e., $u_m \in C^\infty(\bar{U})$.

A.2.3 Sobolev Embeddings and Poincaré Inequalities

The first embedding theorem follows from the Gagliardo-Nirenberg-Sobolev inequality.

Theorem A.18 (Sobolev embedding 1)**.** *Let U be a bounded open subset of $\mathbb{R}^n$ and suppose ∂U is C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$ with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C(n,p,U)\|u\|_{W^{1,p}(U)},$$

where the constant $C = C(n,p,U)$ depends only on $n, p,$ and U.

Proof. Since ∂U is C^1, the extension theorem of Theorem A.13 implies that there exists an extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$ such that $\bar{u} = u$ in U, $\bar{u}$ has compact support, and

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}. \tag{A.15}$$

Since $\bar{u} \in W^{1,p}(\mathbb{R}^n)$ has compact support, the Density theorem or Theorem A.17 implies that there exists a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ $(m = 1, 2, \ldots)$ such that

$$u_m \longrightarrow \bar{u} \text{ in } W^{1,p}(\mathbb{R}^n). \tag{A.16}$$

From the Gagliardo–Nirenberg–Sobolev inequality, we obtain

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}$$

for all $l, m \geq 1$. Hence,

$$u_m \longrightarrow \bar{u} \text{ in } L^{p^*}(\mathbb{R}^n). \tag{A.17}$$

Moreover, the Gagliardo–Nirenberg–Sobolev inequality also implies

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m\|_{L^p(\mathbb{R}^n)},$$

Therefore, (A.16) and (A.17) imply

$$\|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|D\bar{u}\|_{L^p(\mathbb{R}^n)},$$

This inequality and (A.15) complete the proof. □

Theorem A.19 (Sobolev embedding 2)**.** *Assume U is a bounded open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then we have the estimate*

$$\|u\|_{L^q(U)} \leq C(n, p, q, U)\|Du\|_{L^p(U)}$$

for each $q \in [1, p^*]$, where the constant $C = C(n, p, q, U)$ depends only on n, p, q, and U. In particular, for all $1 \leq p \leq \infty$,

$$\|u\|_{L^p(U)} \leq C(n, p, q, U)\|Du\|_{L^p(U)}. \tag{A.18}$$

Remark A.3. *Estimate* (A.18) *is sometimes called Poincaré's inequality. Consequently, this inequality implies the norm $\|Du\|_{L^p(U)}$ is equivalent to $\|u\|_{W^{1,p}(U)}$ in $W_0^{1,p}(U)$ provided U is bounded.*

Proof of Theorem A.19. Since $u \in W_0^{1,p}(U)$, there exist functions $u_m \in C_c^\infty(U)$ $(m = 1, 2, \ldots)$ converging to u in $W^{1,p}(U)$. We extend each function u_m to be 0 on $\mathbb{R}^n \backslash \bar{U}$ (we do not need to invoke the extension theorem) and apply the Gagliardo–Nirenberg–Sobolev inequality to obtain

$$\|u\|_{L^{p^*}(U)} \leq C\|Du\|_{L^p(U)}.$$

Since $\mu(U) < \infty$, basic interpolation theory says the identity map, $I : L^{p^*}(U) \longrightarrow L^q(U)$, is bounded provided $1 \leq q \leq p^*$, i.e., $\|u\|_{L^q(U)} \leq C\|u\|_{L^{p^*}(U)}$ if $1 \leq q \leq p^*$. □

Definition A.5. *We say u^* is a version of a given function u if $u = u^*$ a.e.*

The next embedding theorem is a result of Morrey's inequality.

Theorem A.20 (Sobolev embedding 3)**.** *Let U be a bounded open subset of $\mathbb{R}^n$ and suppose its boundary ∂U is C^1. Assume $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\bar{U})$, for $\gamma = 1 - \frac{n}{p}$, with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leq C(n,p,U)\|u\|_{W^{1,p}(U)}.$$

The constant $C = C(n,p,U)$ depends only on n, p and U.

Proof. We only consider the case $n < p < \infty$ since the case $p = \infty$ is easy to prove directly. Since ∂U is C^1, the extension theorem implies that there is an extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$ such that $\bar{u} = u$ in U, $\bar{u}$ has compact support, and

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}. \tag{A.19}$$

Since $\bar{u}$ has compact support, Theorem A.17 implies there exist functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that

$$u_m \longrightarrow \bar{u} \text{ in } W^{1,p}(\mathbb{R}^n). \tag{A.20}$$

According to Morrey's inequality, $\|u_m - u_l\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u_m - u_l\|_{W^{1,p}(\mathbb{R}^n)}$ where $\gamma = 1 - \frac{n}{p}$ for all $l, m \geq 1$. Hence, there exists a function $u^* \in C^{0,\gamma}(\mathbb{R}^n)$ such that

$$u_m \longrightarrow u^* \text{ in } C^{0,\gamma}(\mathbb{R}^n). \tag{A.21}$$

Owing to (A.20) and (A.21), we see that $u = u^*$ a.e. in U, so u^* is a version of u. Morrey's inequality also implies $\|u_m\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u_m\|_{W^{1,p}(\mathbb{R}^n)}$. Thus, (A.20) and (A.21) imply

$$\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)}.$$

This inequality and (A.19) complete the proof of the theorem. □

The previous Sobolev inequalities for $W^{1,p}(U)$ can be further generalized to the Sobolev spaces $W^{k,p}(U)$ for $k \in \mathbb{N}$.

Theorem A.21 (General Sobolev inequalities)**.** *Let U be a bounded open subset of $\mathbb{R}^n$ with a C^1 boundary ∂U. Assume $u \in W^{k,p}(U)$.*

(i) If $k < \frac{n}{p}$, then $u \in L^q(U)$ where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n} \Longleftrightarrow q = \frac{np}{n - kp}.$$

We have, in addition, the estimate

$$\|u\|_{L^q(U)} \leq C(k,n,p,U)\|u\|_{W^{k,p}(U)}.$$

The constant $C = C(k,n,p,U)$ depends only on $k, n, p,$ and U.

(ii) If $k > \frac{n}{p}$, then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})$, where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & if\ \frac{n}{p} \text{ is not an integer },\\ \text{any positive number } < 1, & if\ \frac{n}{p} \text{ is an integer.} \end{cases}$$

We have, in addition, the estimate

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})} \leq C(k,n,p,\gamma,U)\|u\|_{W^{k,p}(U)},$$

the constant $C = C(k,n,p,\gamma,U)$ depending only on k,n,p,γ, and U.

Proof. The proof is standard, similar to the aforementioned special cases above, and we refer the reader to Evans [9] for more details. □

Remark A.4 (Case $p = n$)**.** *In the endpoint borderline case for $p \in [1,n)$, $p^* = np/(n-p) \longrightarrow +\infty$ by sending $p \longrightarrow n$ which suggests that $W^{1,n}(U) \subset L^\infty(U)$. Unfortunately, this only holds when $n = 1$ and fails for $n \geq 2$. For example, if we take $n \geq 2$ and $U = B_1(0) \subset \mathbb{R}^n$, then the function $\log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,n}(U)$ but not to $L^\infty(U)$. However, $BMO(U)$, the space of functions with bounded mean oscillation, is the proper embedding space to replace $L^\infty(U)$ in order to preserve the embedding of the Sobolev space (see Corollary A.1).*

The next theorem is on the compact embedding of Sobolev spaces into Lebesgue spaces.

Theorem A.22 (Rellich–Kondrachov compactness)**.** *Assume U is a bounded open subset of $\mathbb{R}^n$ with C^1 boundary ∂U. Suppose $1 \leq p < n$, then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for each $1 \leq q < p^*$.

Proof. **1.** Fix $1 \leq q < p^*$ and note that since U is bounded, Theorem A.18 implies $W^{1,p}(U) \subset L^q(U)$ and $\|u\|_{L^q(U)} \leq C\|u\|_{W^{1,p}(U)}$. Thus, it remains to show that if $\{u_m\}_{m=1}^\infty$ is a bounded sequence in $W^{1,p}(U)$, there exists a subsequence $\{u_{m_j}\}_{j=1}^\infty$ which converges in $L^q(U)$.
2. By the Extension theorem, we may assume, without loss of generality, that $U = \mathbb{R}^n$ and the functions $\{u_m\}_{m=1}^\infty$ all have compact support in some bounded open set $V \subset \mathbb{R}^n$. We also may assume

$$\sup_m \|u_m\|_{W^{1,p}(U)} < \infty. \tag{A.22}$$

3. We first examine the smoothed functions

$$u_m^\epsilon := \eta_\epsilon * u_m \ (\epsilon > 0, m = 1,2,3,\ldots),$$

where η_ϵ denotes the standard mollifier. We may assume that the functions $\{u_m^\epsilon\}_{m=1}^\infty$ all have support in V as well.

4. We claim that

$$u_m^\epsilon \longrightarrow u_m \text{ in } L^q(V) \text{ as } \epsilon \longrightarrow 0 \text{ uniformly in } m. \tag{A.23}$$

To prove this, we note that if u_m is smooth, then

$$\begin{aligned} u_m^\epsilon(x) - u_m(x) &= \frac{1}{\epsilon^n} \int_{B_\epsilon(x)} \eta\left(\frac{x-z}{\epsilon}\right)(u_m(z) - u_m(x))\,dz \\ &= \int_{B_1(0)} \eta(y)(u_m(x-\epsilon y) - u_m(x))\,dy \\ &= \int_{B_1(0)} \eta(y) \int_0^1 \frac{d}{dt}(u_m(x-\epsilon ty))\,dt\,dy \\ &= -\epsilon \int_{B_1(0)} \eta(y) \int_0^1 Du_m(x-\epsilon ty)\cdot y\,dt\,dy. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_V |u_m^\epsilon(x) - u_m(x)|\,dx &\le \epsilon \int_{B_1(0)} \eta(y) \int_0^1 \int_V |Du_m(x-\epsilon ty)|\,dx\,dt\,dy \\ &\le \epsilon \int_V |Du_m(z)|\,dz. \end{aligned}$$

By approximation, this estimate holds if $u_m \in W^{1,p}(V)$. Since V is bounded, we obtain

$$\|u_m^\epsilon - u_m\|_{L^1(V)} \le \epsilon \|Du_m\|_{L^1(V)} \le \epsilon C \|Du_m\|_{L^p(V)},$$

By virtue of (A.22), we have

$$u_m^\epsilon \longrightarrow u_m \text{ in } L^1(V) \text{ uniformly in } m. \tag{A.24}$$

Then since $1 \le q < p^*$, the L^p interpolation inequality yields

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \le \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta \|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{1-\theta},$$

where $\frac{1}{q} = \theta + \frac{(1-\theta)}{p^*}$ and $\theta \in (0,1)$. As a consequence of (A.22) and the Gagliardo–Nirenberg–Sobolev inequality, we obtain

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \le C \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta.$$

Hence, (A.23) follows from (A.22).
5. Next, we claim that for each $\epsilon > 0$, the sequence $\{u_m\}_{m=1}^\infty$ is uniformly bounded and equicontinuous.

Indeed, if $x \in \mathbb{R}^n$, then

$$|u_m^\epsilon(x)| \leq \int_{B_\epsilon(x)} \eta_\epsilon(x-y)|u_m(y)|\,dy \leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \leq C\epsilon^{-n} < \infty,$$

for $m = 1, 2, \ldots$. Similarly,

$$|Du_m^\epsilon(x)| \leq \int_{B_\epsilon(x)} |D\eta_\epsilon(x-y)||u_m(y)|\,dy \leq \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \leq C\epsilon^{-(n+1)} < \infty,$$

for $m = 1, 2, \ldots$. Thus, the claim follows from these two estimates.
6. Now fix $\delta > 0$. We show that there exists a subsequence $\{u_{m_j}\}_{j=1}^\infty \subset \{u_m\}_{m=1}^\infty$ such that

$$\lim_{j,k\longrightarrow\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta. \tag{A.25}$$

To see this, we employ (A.23) to select $\epsilon > 0$ suitably small such that

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \delta/2 \tag{A.26}$$

for $m = 1, 2, \ldots$.

Now observe that since the functions $\{u_m\}_{m=1}^\infty$, and thus the functions $\{u_m^\epsilon\}_{m=1}^\infty$, have support in some fixed bounded set $V \subset \mathbb{R}^n$, we can apply the claim in 5. and the Arzelà–Ascoli compactness theorem to extract a subsequence $\{u_{m_j}^\epsilon\}_{j=1}^\infty \subset \{u_m^\epsilon\}_{m=1}^\infty$ which converges uniformly on V. Therefore,

$$\limsup_{j,k\longrightarrow\infty} \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\|_{L^q(V)} = 0.$$

But then this combined with (A.26) imply

$$\limsup_{j,k\longrightarrow\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta.$$

This proves (A.25).
7. By applying assertion (A.25) with $\delta = 1, 1/2, 1/3, \ldots$ and use a standard diagonal argument to extract a subsequence $\{u_{m_j}\}_{j=1}^\infty \subset \{u_m\}_{m=1}^\infty$ satisfying

$$\limsup_{l,k\longrightarrow\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0.$$

This completes the proof of the theorem. □

Remark A.5. *Since $p^* > p$ and $p^* \longrightarrow \infty$ as $p \longrightarrow n$, we have*

$$W^{1,p}(U) \subset\subset L^p(U)$$

for all $1 \leq p \leq \infty$. In addition, note that

$$W_0^{1,p}(U) \subset\subset L^p(U),$$

even if we do not assume ∂U is C^1.

The Rellich–Kondrachov compactness theorem allows us to establish the following Poincaré type inequalities. We omit their proofs but refer the readers to Evans [9] for more details.

Theorem A.23 (Poincaré's inequality)**.** *Let U be a bounded, connected, and open subset of $\mathbb{R}^n$ with C^1 boundary ∂U. Assume $1 \le p \le \infty$. Then there exists a constant $C = C(n,p,U)$ depending only on n, p, and U, such that*

$$\|u - (u)_U\|_{L^p(U)} \le C(n,p,U)\|Du\|_{L^p(U)}$$

for each function $u \in W^{1,p}(U)$ where $(u)_U := \frac{1}{|U|}\int_U u\,dy$.

Theorem A.24 (Poincaré's inequality on balls)**.** *Assume $1 \le p \le \infty$. Then there exists a constant $C = C(n,p)$ depending only on n and p such that*

$$\|u - (u)_{x,r}\|_{L^p(B_r(x))} \le C(n,p) \cdot r\|Du\|_{L^p(B_r(x))}$$

for each ball $B_r(x) \subset \mathbb{R}^n$ and each function $u \in W^{1,p}(B_r(x))$ where $(u)_{x,r} := \frac{1}{|B_r(x)|}\int_{B_r(x)} u\,dy$.

A simple application is the embedding of $W^{1,p}(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$.

Corollary A.1. *Let $n \ge 1$ and suppose $u \in W^{1,n}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then $u \in BMO(\mathbb{R}^n)$.*

Proof. From Theorem A.24 with $p = 1$ and Hölder's inequality, we get

$$\begin{aligned}\frac{1}{|B_r(x)|}\int_{B_r(x)} |u - (u)_{x,r}|\,dy &\le Cr\frac{1}{|B_r(x)|}\int_{B_r(x)} |Du|\,dy \\ &\le Cr\Big(\frac{1}{|B_r(x)|}\int_{B_r(x)} |Du|^n\,dy\Big)^{1/n} \\ &\le C\Big(\int_{B_r(x)} |Du|^n\,dy\Big)^{1/n}.\end{aligned}$$

Hence, we deduce that

$$\|u\|_{BMO(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n,\,r>0}\frac{1}{|B_r(x)|}\int_{B_r(x)} |u - (u)_{x,r}|\,dy \le C(n)\|u\|_{W^{1,n}(\mathbb{R}^n)}.$$

□

A.3 Integration and Convergence Theorems

Let $(X, \mathcal{A}, \mu)$ be a given measure space.

Theorem A.25 (Lebesgue's Montone Convergence). *Let $\{f_n\}$ be a* **monotone increasing** *sequence of non-negative measurable functions that converges* **pointwise** *to a function $f(x)$, i.e.,*

(a) $0 \leq f_1(x) \leq f_2(x) \leq \ldots \leq f_n(x) \leq \ldots \leq \infty$ *for every $x \in X$ (monotone increasing),*

(b) and

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \textit{for every } x \in X \textit{ (pointwise convergence).}$$

Then f is measurable and

$$\int_X f_n \, d\mu \longrightarrow \int_X f \, d\mu \quad \textit{as } n \longrightarrow \infty. \tag{A.27}$$

Proof. Since the pointwise limit of a sequence of non-negative measurable functions is also a non-negative measurable function, the limiting function $f : X \longrightarrow [0, \infty]$ is also measurable. Moreover, since $f_n \leq f_{n+1} \leq f$ for all $n \in \mathbb{N}$, we deduce that

$$\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu \leq \int_X f \, d\mu \quad \text{for all } n \in \mathbb{N}.$$

Thus,

$$\lim_{n\to\infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu. \tag{A.28}$$

To obtain the reverse inequality, we choose an arbitrary $0 < \alpha < 1$ and let φ be any simple function satisfying $0 \leq \varphi \leq f$. Set

$$A_n := \Big\{ x \in X \mid f_n(x) \geq \alpha\varphi(x) \Big\}.$$

It is easy to see that $A_n \in \mathcal{A}$ and $A_n \subset A_{n+1}$ for each n, and that $X = \cup_{n=1}^{\infty} A_n$. Under these conditions on $\{A_n\}_{n=1}^{\infty}$, it is a fairly standard exercise to prove that

$$\nu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \nu(A_n)$$

for any measure ν on $(X, \mathcal{A})$, and it is also standard to show $\nu(E) := \int_E \varphi \, d\mu$ indeed provides such a particular measure (actually, we will soon show in Corollary A.3 that this remains valid if φ is any non-negative measurable function and not just a simple function). These standard results imply that

$$\int_X \varphi \, d\mu = \nu(X) = \nu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \nu(A_n) = \lim_{n\to\infty} \int_{A_n} \varphi \, d\mu.$$

On the other hand, by our construction of $\{A_n\}_{n=1}^{\infty}$, we have that

$$\alpha \int_{A_n} \varphi \, d\mu = \int_{A_n} \alpha\varphi \, d\mu \leq \int_{A_n} f_n \, d\mu \leq \int_X f_n \, d\mu.$$

Sending $n \longrightarrow \infty$ in the previous inequality leads us to

$$\alpha \int_X \varphi \, d\mu = \lim_{n\to\infty} \int_{A_n} \alpha\varphi \, dx \leq \lim_{n\to\infty} \int_X f_n \, d\mu. \tag{A.29}$$

Recall that the integral of f is defined by

$$\int_X f \, d\mu = \sup \int_X \phi \, d\mu,$$

where the supremum is taken over all simple functions ϕ such that $0 \leq \phi \leq f$. Therefore, since $0 < \alpha < 1$ and φ were chosen arbitrarily and because of (A.29), we must have that

$$\int_X f \, d\mu \leq \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Combining this with (A.28) completes the proof of the theorem. □

Remark A.6. *The integral in* (A.27) *is allowed to equal* $+\infty$. *Moreover, an analogue result involving non-increasing sequences of functions holds true. Namely, if there is a sequence* $\{f_n\}_{n=1}^{\infty}$ *of non-negative measurable functions with* $f_1 \in L^1(\mu)$ *and this sequence is non-decreasing, i.e.,* $f_1(x) \geq f_2(x) \geq \ldots \geq f(x)$ *for all* $x \in X$*, and if* $f_n(x) \longrightarrow f(x)$ *for all* $x \in X$*, then* $f : X \longrightarrow [0, \infty]$ *is measurable and*

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

The assumption that $f \in L^1(\mu)$ *cannot be omitted.*

We recall several important applications and consequences of the Monotone Convergence Theorem. The first is Fatou's lemma.

Lemma A.1 (Fatou's)**.** *If* $f_n : X \longrightarrow [0, \infty]$ *is measurable for each positive integer* n*, then*

$$\int_X \left(\liminf_{n\to\infty} f_n \right) d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Proof. Set $g_n = \inf_{k\geq n} f_k$ and set $g := \liminf_{n\to\infty} f_n$. Then $g_n : X \longrightarrow [0, \infty]$ is measurable, $g_n \longrightarrow g$ pointwise everywhere in X and $\{g_n\}$ is monotone increasing. By the Monotone Convergence Theorem and the fact that $f_n \geq g_n$ in X, for all n, we obtain

$$\int_X \left(\liminf_{n\to\infty} f_n \right) d\mu = \int_X g \, d\mu = \lim_{n\to\infty} \int_X g_n \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

□

The next is a consequence of Fatou's lemma which we often use. For instance, it implies that strong solutions of elliptic equations on a bounded domain satisfy the equation pointwise almost everywhere in the domain.

Corollary A.2. *Suppose that f is a non-negative measurable function. Then $f = 0$ μ-almost everywhere in X if and only if*

$$\int_X f \, d\mu = 0. \tag{A.30}$$

Proof. If (A.30) holds, let

$$E_n = \left\{ x \in X \,\middle|\, f(x) > 1/n \right\},$$

so that E_n is measurable and $f \geq (1/n)\chi_{E_n}$, from which we see

$$0 = \int_X f \, d\mu \geq \frac{1}{n}\mu(E_n) \geq 0.$$

Thus, $\mu(E_n) = 0$ and so the set

$$\{x \in X \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$$

is measurable and has measure zero by the countable additive property of measures. This verifies that $f = 0$ μ-almost everywhere in X.

Conversely, assume $f = 0$ μ-almost everywhere. If

$$E = \{x \in X \mid f(x) > 0\},$$

then obviously E is measurable with $\mu(E) = 0$. Then set $f_n = n\chi_E$ so that each f_n is non-negative and measurable, and clearly $f \leq \liminf_{n\to\infty} f_n$. Thus, by Fatou's lemma,

$$0 \leq \int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu = \liminf_{n\to\infty} n\mu(E) = 0.$$

Hence, $\|f\|_{L^1(\mu)} = 0$, and this completes the proof. □

The next consequence illustrates we can use the integral of any non-negative measurable function to construct another measure that is absolutely continuous with respect to the original measure.

Corollary A.3. *If $f : X \longrightarrow [0, \infty]$ is a non-negative measurable function and if λ is defined on the σ-algebra $\mathcal{A}$ by*

$$\lambda(E) = \int_E f \, d\mu, \tag{A.31}$$

then λ is a measure on the measurable space $(X, \mathcal{A})$. Moreover, the measure λ is absolutely continuous with respect to μ in the sense that if $E \in \mathcal{A}$ and $\mu(E) = 0$, then $\lambda(E) = 0$.

Proof. We verify λ defines a measure. Obviously, $\lambda(\emptyset) = 0$. Now, suppose that $\{E_n\}_{n=1}^{\infty} \subset \mathcal{A}$ is a sequence of disjoint measurable sets. Set $E := \cup_{n=1}^{\infty} E_n \in \mathcal{A}$ and define

$$f_n(x) = \sum_{k=1}^{n} f\chi_{E_k}.$$

Indeed, f_n is a non-negative measurable function and

$$\int_X f_n\,du = \sum_{k=1}^{n} \int_X f\chi_{E_k}\,d\mu = \sum_{k=1}^{n} \lambda(E_k).$$

Then $\{f_n\}$ is a monotone increasing sequence of non-negative, measurable functions converging pointwise to f on X. Hence, the Monotone Convergence Theorem implies that

$$\lambda(E) = \int_E f\,du = \lim_{n\to\infty} \int_X f_n\,d\mu = \sum_{n=1}^{\infty} \lambda(E_n),$$

and therefore λ defines a measure.

Assume now that $E \in \mathcal{A}$ such that $\mu(E) = 0$. The function $f\chi_E$ vanishes μ-almost everywhere. So, by Corollary A.2, we deduce that

$$\lambda(E) = \int_X f\chi_E\,d\mu = 0.$$

□

Remark A.7. *In general, we write $\lambda \ll \mu$ to mean λ is absolutely continuous with respect to μ. Under suitable conditions, the converse of Corollary A.3 holds, and this result is well-known and is referred to as the Radon-Nikodym Theorem. We state it below for completeness but omit its proof. The proof can be found in any standard graduate real analysis textbook, e.g., see [3, 10, 28].*

Theorem A.26 (Radon-Nikodym)**.** *Let λ and μ be σ-finite measures on $(X, \mathcal{A})$ and suppose $\lambda \ll \mu$. Then there exists a measuable function $f : X \longrightarrow [0, \infty]$ such that*

$$\lambda(E) = \int_E f\,d\mu,\ \ E \in \mathcal{A}.$$

Moreover, the function f is uniquely determined μ-almost everywhere.

Remark A.8. *The function f in Theorem A.26 is called the Radon-Nikodym derivative of λ with respect to μ and we write*

$$\frac{d\lambda}{d\mu} = f.$$

We can invoke the earlier corollary to replace pointwise convergence with μ-almost everywhere convergence in Theorem A.25 but the limit function is assumed to be measurable a priori.

Corollary A.4. *Let $\{f_n\}$ be a monotone increasing sequence of non-negative measurable functions that converges μ-almost everywhere in X to a non-negative measurable function $f(x)$. Then*

$$\int_X f\,d\mu = \lim_{n\to\infty}\int_X f_n\,d\mu.$$

Proof. Choose $N \in \mathcal{A}$ be such that $\mu(N) = 0$ and $\{f_n\}$ converges to f at every point of $M = X\backslash N$. Then $\{f_n\chi_M\}$ converges to $f\chi_M$ in X. Thus Theorem A.25 implies that

$$\int_X f\chi_M\,d\mu = \lim_{n\to\infty}\int_X f_n\chi_M\,d\mu.$$

Since $\mu(N) = 0$, the functions $f\chi_N$ and $f_n\chi_N$ vanish μ-almost everywhere. It follows from A.30 that

$$\int_X f\chi_N\,d\mu = 0 \text{ and } \int_X f_n\chi_N\,d\mu = 0.$$

Since $f = f\chi_M + f\chi_N$ and $f_n = f_n\chi_M + f_n\chi_N$, it follows that

$$\int_X f\,d\mu = \int_X f\chi_M\,d\mu = \lim_{n\to\infty}\int_X f_n\chi_M\,d\mu = \lim_{n\to\infty}\int_X f_n\,d\mu.$$

□

An essential convergence theorem often utilized in our applications is Lebesgue's Dominated Convergence Theorem (LDCT). This useful result simply follows from Fatou's lemma and the following basic fact.

Lemma A.2. *If $f \in L^1(\mu)$, then*

$$\Big|\int_X f\,d\mu\Big| \le \int_X |f|\,d\mu.$$

Proof. Set $z = \int_X f\,d\mu \in \mathbb{C}$. Thus, $|z| = \alpha z$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. If $u = Re(\alpha f)$, then $u \le |\alpha f| = |f|$ and so

$$|z| = \alpha z = \alpha\int_X f\,d\mu = \int_X \alpha f\,d\mu = \int_X u\,d\mu \le \int_X |f|\,d\mu,$$

where we used the fact that $\int_X \alpha f\,d\mu$ is real. □

Theorem A.27 (Lebesgue's Dominated Convergence)**.** *Suppose $\{f_n\}$ is a sequence of measurable functions on X such that*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \leq g(x) \quad \text{for } n = 1, 2, 3, \ldots; \; x \in X,$$

then $f \in L^1(\mu)$,

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$$

and

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Proof. Indeed, $f \in L^1(\mu)$, since $|f| \leq g \in L^1(\mu)$ and f is measurable. We similarly deduce that $f_n \in L^1(\mu)$ for all n.

From the triangle inequality, we also get that $|f_n - f| \leq 2g$, and so $f_n - f \in L^1(\mu)$. Applying Fatou's lemma to the non-negative functions $2g - |f_n - f|$ leads us to

$$\begin{aligned}
\int_X 2g \, d\mu &\leq \liminf_{n\to\infty} \int_X (2g - |f_n - f|) \, d\mu \\
&= \int_X 2g \, d\mu + \liminf_{n\to\infty} \Big(- \int_X |f_n - f| \, d\mu \Big) \\
&= \int_X 2g \, d\mu - \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu.
\end{aligned}$$

Noting that $\int_X g \, d\mu$ is finite, we may add $-2 \int_X g \, d\mu$ to previous inequality to arrive at

$$\limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \leq 0.$$

This further implies that

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0. \tag{A.32}$$

Since $f_n - f \in L^1(\mu)$, Lemma A.2 implies that

$$\Big| \int_X (f_n - f) \, d\mu \Big| \leq \int_X |f_n - f| \, d\mu,$$

and so (A.32) further yields that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

□

Remark A.9. *In Theorem A.27, we can easily weaken the statement and only assume that pointwise convergence holds in the μ-almost everywhere sense. This is because we can always redefine f_n and f on a set of measure zero.*

More precisely, since a countable union of measurable sets of measure zero is measurable and also has measure zero, we can find a measurable set E with $\mu(E) = 0$ and redefine $\{f_n\}$, and similarly with f, so that $f_n(x) = 0$ for $x \in E$ and $f_n(x)$ remains unchanged for $x \notin E$. Note this does not change the value of the integrals $\int_X f_n \, d\mu$.

An immediate application of Theorem A.27 is the following

Corollary A.5. *If $t \longrightarrow f(x,t)$ is continuous on $[a,b]$ for each $x \in X$, and if there exists $g \in L^1(\mu)$ such that $|f(x,t)| \le g(x)$ for $x \in X$, then the function F defined by*

$$F(t) = \int_X f(x,t) \, d\mu(x) \tag{A.33}$$

is continuous for each t in $[a,b]$.

Another basic application of Theorem A.27 indicates when we may differentiate F and when it is equivalent to passing derivatives onto the integrand f. Hereafter, an integrable function f on X means f is a measurable function on X belonging to $L^1(\mu)$.

Corollary A.6. *Suppose that for some t_0 in $[a,b]$, the function $x \longrightarrow f(x,t_0)$ is integrable on X, that $\partial f/\partial t$ exists on $X \times [a,b]$, and that there exists an integrable function g on X such that*

$$\left| \frac{\partial f}{\partial t}(x,t) \right| \le g(x).$$

Then the function F as defined in (A.33) is differentiable on $[a,b]$ and

$$\frac{dF}{dt}(t) = \frac{d}{dt} \int_X f(x,t) \, d\mu(x) = \int_X \frac{\partial f}{\partial t}(x,t) \, d\mu(x).$$

Proof. Let t be any point of $[a,b]$. If $\{t_n\}$ is a sequence in $[a,b]$ converging to t with $t_n \neq t$, then

$$\frac{\partial f}{\partial t}(x,t) = \lim_{n \to \infty} \frac{f(x,t_n) - f(x,t)}{t_n - t}, \ x \in X.$$

Therefore, the function $x \longrightarrow (\partial f/\partial t)(x,t)$ is measurable.

If $x \in X$ and $t \in [a,b]$, by the mean-value theorem, there exists s_1 between t_0 and t such that

$$f(x,t) - f(x,t_0) = (t - t_0) \frac{\partial f}{\partial t}(x, s_1).$$

Therefore,

$$|f(x,t)| \le |f(x,t_0)| + |t - t_0| g(x),$$

which implies that the function $x \longrightarrow f(x,t)$ is integrable for each t in $[a,b]$. Hence, if $t_n \neq t$, then

$$\frac{F(t_n) - F(t)}{t_n - t} = \int_X \frac{f(x,t_n) - f(x,t)}{t_n - t}\, d\mu(x).$$

Since this integrand is dominated by $g(x)$, we may apply Theorem A.27 to conclude the desired result.

□

We can use Theorem A.27 to establish a similar convergence result in the Lebesgue spaces $L^p(\mu)$ with $1 \leq p < \infty$.

Theorem A.28. *Let $1 \leq p < \infty$ and suppose $\{f_n\}$ is a sequence in $L^p(\mu)$ which converges μ-almost everywhere to a measurable function f. If there exists a $g \in L^p(\mu)$ such that*

$$|f_n(x)| \leq g(x), \ x \in X, \ n \in N,$$

then f belongs to $L^p(\mu)$ and $\{f_n\}$ converges in $L^p(\mu)$ to f.

Proof. Assume $1 < p < \infty$ since the case $p = 1$ is exactly Theorem A.27. Obviously, the following two properties hold for μ-almost everywhere,

$$|f_n(x) - f(x)|^p \leq [2g(x)]^p, \ \text{ and } \ \lim_{n\to\infty} |f_n(x) - f(x)|^p = 0;$$

and there holds $[2g]^p$ and thus g^p belongs to $L^1(\mu)$. Hence, from Theorem A.27, we get

$$\lim_{n\to\infty} \int_X |f_n - f|^p \, d\mu = 0,$$

and this completes the proof of the theorem.

□

Remark A.10. *Lebesgue's dominated convergence theorem and its extension provide sufficient conditions that guarantee when pointwise convergence of a sequence of measurable functions implies strong convergence in the L^p norm topology; namely, if the sequence of functions can be compared to an L^p function, then pointwise convergence implies L^p convergence. Conversely, L^p convergence does not generally imply pointwise convergence. We give an example below illustrating this.*

Let $X = [0,1]$, the sigma algebra $\mathcal{A}$ are the Borel sets, and μ is the Lebesgue measure. Consider the ordered list of intervals

$[0,1], [0,\frac{1}{2}], [\frac{1}{2},1], [0,\frac{1}{3}], [\frac{1}{3},\frac{2}{3}], [\frac{2}{3},1], [0,\frac{1}{4}], [\frac{1}{4},\frac{1}{2}], [\frac{1}{2},\frac{3}{4}], [\frac{3}{4},1], [0,\frac{1}{5}], [\frac{1}{5},\frac{2}{5}], \ldots$; let f_n be the characteristic function of the n^{th} interval on this list, and let f be identically zero. If $n > m(m+1)/2 = 1+2+\ldots+m$, then f_n is a characteristic function of an interval I whose measure is at most $1/m$. Hence,

$$\|f_n - f\|^p_{L^p(\mu)} = \int_X |f_n - f|^p \, d\mu = \int_X |f_n|^p \, d\mu = \int_X f_n \, d\mu = \mu(I) \leq 1/m,$$

and this shows $\{f_n\}$ converges in L^p to $f \equiv 0$.

On the other hand, if x is any point of $[0,1]$, then the sequence of numbers $\{f_n(x)\}$ has a subsequence consisting only of 1's and another subsequence consisting of 0's. Therefore, the sequence $\{f_n\}$ does not converge at any point of $[0,1]$! (although we may select a particular subsequence of $\{f_n\}$ which does converge to f).

The next result swaps the domination condition in the LDCT with finite measure and uniform integrability.

Theorem A.29 (Vitali's Convergence Theorem). *Let $(X, \mathcal{A}, \mu)$ be of finite measure, i.e., $\mu(X) < +\infty$, and suppose the sequence $\{f_n\}$ is uniformly integrable over X, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that for each n, E measurable and $\mu(E) < \delta$ implies $\int_E |f_n| \, d\mu < \epsilon$. If $\{f_n\}$ converges pointwise μ-a.e. in X to f, then*

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

APPENDIX B

A brief introduction to smooth manifolds and Riemannian geometry

Many of the model problems and applications studied in this textbook arise from fundamental questions from topology and geometry. These problems were chosen carefully to illustrate how the theory of elliptic PDEs play a prominent role in their complete resolution. For convenience and in our attempt to keep the presentation of topics in this book self-contained, we provide some essential definitions, examples and results from smooth manifold theory and Riemannian geometry.

B.1 Topological Manifolds; Smooth Manifolds

First, we define the notion of a topological manifold, which are, locally speaking, just like the Euclidean spaces, and they may be viewed as the higher-dimensional analogues of curves and surfaces in space.

Definition B.1. *A topological space M is* **locally Euclidean** *of dimension $n \in \mathbb{N}$ if every point $x \in M$ has a neighborhood U such that there is a homeomorphism $\phi : U \longrightarrow \mathbb{R}^n$ from U onto an open subset $\phi(U)$ of $\mathbb{R}^n$. We call the pair (U, ϕ) a chart, U a coordinate neighborhood or a coordinate open set, and ϕ a coordinate map or coordinate system on U. For $x \in U$, we call $\phi(x)$ the coordinates of x. And we say (U, ϕ) is centered at $x \in U$ if $\phi(x) = 0$.*

To define a topological manifold, and therefore a smooth manifold, we shall restrict our attention to Hausdorff and second countable spaces. Recall, we say a sub-collection $\mathcal{B} = \{B_\alpha\}$ of a topology τ for a topological space (M, τ) is a basis for the topology τ if given an open set U and any point $p \in U$, there is a $B_{\alpha'} \in \mathcal{B}$. Then, we say a topological space M is **second countable** if it has a countable basis. Moreover, we say a topological space M

is **Hausdorff** (or τ_2) if for each distinct pair $x, y \in M$, there exist disjoint open sets U_x and U_y such that $x \in U_x$ and $y \in U_y$.

Definition B.2. *A* **topological manifold** *M is a Hausdorff, second countable, locally Euclidean space. We say M is of dimension n if it is locally Euclidean of dimension n, and we sometimes write M^n in place of M.*

The reason for restricting to Hausdorff spaces is to avoid some pathological examples, and the second countability assumption ensures the existence of a partition of unity, a useful tool in studying manifolds.

Definition B.3. *Let $M = M^n$ be a topological manifold of dimension n with two charts (U, ϕ) and (V, ψ). We say the pair of charts is compatible if the two maps*

$$\phi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \phi(U \cap V) \text{ and } \psi \circ \phi^{-1} : \phi(U \cap V) \longrightarrow \psi(U \cap V)$$

are of the class C^∞. Moreover, a C^∞ atlas, or simply an atlas, on M is a collection $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ of pairwise C^∞-compatible charts that cover M, i.e., $M = \cup_{\alpha \in \mathcal{A}} U_\alpha$.

We say a chart (V, ψ) is compatible with an atlas $\{(U_\alpha, \phi_\alpha)\}$ if it is compatible with all the charts in the atlas. In fact, it turns out that if two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then they must be compatible with each other.

Definition B.4. *A smooth manifold is a topological manifold M together with a maximal atlas (by maximal we mean it is not contained in a strictly larger C^∞ atlas). This maximal atlas is sometimes referred as a differentiable structure on M. A manifold is said to have dimension n if all of its connected components have dimension n. Specifically, a 1-dimensional manifold is also called a curve, a 2-dimensional manifold a surface, and an n-dimensional manifold an n-manifold.*

To determine if a topological manifold M is indeed a smooth manifold, it is not necessary to find a maximal atlas. The following proposition illustrates that it is enough to determine the existence of any atlas on M.

Proposition B.1. *Any atlas $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ on a locally Euclidean space is contained in a unique maximal atlas.*

Unless otherwise stated, whenever we say manifold, it should be understood that we always mean a smooth manifold. Perhaps, the canonical example of an n dimensional smooth manifold is the Euclidean space itself, $\mathbb{R}^n$, endowed with the usual inner product. Other classical examples are the real projective space, the unit sphere, and the torus.

Example: The real projective space.

Example: The standard unit sphere.

Example: The n-torus.

To define the notion of a diffeomorphism and when to interepret two smooth manifolds are diffeomorphic, i.e., equivalent, we need to make precise the meaning of a differentiable function between smooth manifolds. So, given a pair of smooth manifolds $M = M^m$ and $N = N^n$, let $f : M \longrightarrow N$ is some mapping from M to N. Then, f is **differentiable** (or of class C^k) if for all charts (U, φ) and $(\tilde{U}, \tilde{\phi})$ of M and N, respectively, such that $f(U) \subset \tilde{U}$, the map

$$\tilde{\phi} \circ f \circ \phi^{-1} : \phi(U) \longrightarrow \tilde{\phi}(\tilde{U})$$

is differentiable (or of class C^k) in the classical sense the respective regularity holds as a mapping in Euclidean space. Thus, a smooth or C^∞ mapping $f : M \longrightarrow N$ is said to be a **diffeomorphism** if its inverse exists and is smooth. If such a diffeomorphism exists, then the manifolds M and N are said to be **diffeomorphic**.

We further define the rank $R(f)_x$ of f at a point $x \in M$ as the rank of $\tilde{\phi} \circ f \circ \phi^{-1}$ at $\phi(x) \in \mathbb{R}^m$, where the charts (U, φ) and $(\tilde{U}, \tilde{\phi})$ are defined as above but with the additional property that $x \in U$. Now, this definition of $R(f)_x$ is an intrinsic property in that it does not depend on the choice of the charts, but this is left to the reader to check. Then, the map f is said to be an **immersion** if, for all $x \in M$, $R(f)_x = m = dim(M)$, and and it is a **submersion** if for any $x \in M$, $R(f)_x = n = dim(N)$. Furthermore, it is said to be an **embedding** if it is an immersion that realizes a homeomoprhism onto its image.

B.2 Tangent planes and tangent bundles

Again, we take M to be a smooth manifold and let $x \in M$. Denote by $\mathcal{F}_x$ to be the real vector space of functions $f : M \longrightarrow \mathbb{R}$ that are differentiable at x. We say a $f \in \mathcal{F}_x$ is **flat at** x if there exists a chart (U, ϕ) at x such that $D(f \circ \phi^{-1})_{\phi(x)} = 0$, and we define $\mathcal{N}_x$ to be the vector subspace of such functions. A linear form X on $\mathcal{F}_x$ is said to be a **tangent vector** of M at x if $\mathcal{N}_x \subset ker(X)$. In other words, a linear functional $X : \mathcal{F}_x \longrightarrow \mathbb{R}$ in the dual space $\mathcal{F}_x^*$ is a tangent vector at x if $X(f) = 0$ for all flat functions $f \in \mathcal{N}_x$.

Definition B.5. *The set of all tangent vectors to M at a point $x \in M$ is called the tangent space of M at x and will be denoted by $T_x(M)$ or T_xM. The* **tangent bundle** *of M, denoted by or $T(M)$ or TM, is defined as the disjoint union of the tangent spaces T_xM over all $x \in M$.*

Given some chart (U, ϕ) of $x \in M$, with associated coordinates x^i, we define the tangent vectors $\left(\frac{\partial}{\partial x_i}\right)_x \in T_xM$ by

$$\left(\frac{\partial}{\partial x_i}\right)_x \cdot (f) = D(f \circ \phi^{-1})_{\phi(x)} \text{ for each } f \in \mathcal{F}_x.$$

It is a straightforward exercise to show $\left(\frac{\partial}{\partial x_i}\right)_x$'s form a basis for T_xM.

If M is n dimensional, then there is a natural smooth structure we can place on the tangent bundle TM making into a $2n$-dimensional smooth manifold. More precisely, given a chart (U, ϕ) of M, one can show

$$\Big(\bigcup_{x\in U} T_xM, \Phi\Big)$$

forms a chart of TM, where, for $X \in T_xM$, $x \in U$,

$$\Phi(X) = \big(\phi^1(x), \phi^2(x), \ldots, \phi^n(x), X(\phi^1), X(\phi^2), \ldots, X(\phi^n)\big).$$

Equipped with this notion of a tangent bundle, we define a **vector field** on a smooth manifold M as a mapping $X : M \longrightarrow TM$ such that for any $x \in M$, $X(x) \in T_xM$. And since M and its tangent bundle TM are smooth manifolds, we can make sense of vector fields of class C^k.

Definition B.6. *Suppose that M and N are two smooth manifolds, $x \in M$, and $f : M \longrightarrow N$ is differentiable at x. The* **differential map** *of f at x, denoted by $f_*(x)$, is the linear map from T_xM to $T_{f(x)}N$ such that for each tangent vector $X \in T_xM$, the differential map evaluated at X is defined by*

$$(f_*(x)(X))(g) = X(g \circ f) \quad \textit{for every } g : N \longrightarrow \mathbb{R} \textit{ differentiable at } f(x).$$

More generally, if f is differentiable on M, the **differential map** *of f, denoted by $f_* : TM \longrightarrow TN$, such that*

$$f_*(X) = f_*(x)(X) \quad \textit{for } X \in T_xM.$$

Remark B.1. *One can check that $f_* \in C^{k-1}$ whenever $f \in C^k$; and for $f : M_1 \longrightarrow M_2$ and $g : M_2 \longrightarrow M_3$, and $x \in M_1$, there holds $(g \circ f)_*(x) = g_*(f(x)) \circ f_*(x)$.*

Similar to the construction of the tangent bundle, we define the cotangent bundle of a smooth manifold by duality. For $x \in M$, let T_xM^* be the dual space of T_xM. If (U, ϕ) is a chart of M at x of associated coordinates x^i, we obtain a basis for T_xM^* by considering dx^i_x for $i = 1, 2, \ldots, n$, where

$$dx^i_x \cdot \Big(\frac{\partial}{\partial x_j}\Big)_x = \delta^i_j. \tag{B.1}$$

Then, the **cotangent bundle** of M, denoted by $T^*(M)$ or T^*M, is the disjoint union of T_xM^* over all $x \in M$. The cotangent bundle possesses a natural structure making it into a $2n$-dimensional smooth manifold. Given a chart (U, ϕ) of M,

$$\Big(\bigcup_{x\in U} T_xM^*, \Phi\Big)$$

is a chart of T^*M, where for $\eta \in T_xM^*$, $x \in U$,

$$\Phi(\eta) = \Big(\phi^1(x), \phi^2(x), \ldots, \phi^n(x), \eta\Big(\frac{\partial}{\partial x_1}\Big), \eta\Big(\frac{\partial}{\partial x_2}\Big), \ldots, \eta\Big(\frac{\partial}{\partial x_n}\Big)\Big).$$

B.3 Differential Forms

In this section, suppose M is an n-dimensional smooth manifold.

For an integer $q \geq 1$, a **q-form** or a **form of degree** q on M is the assignment to each $x \in M$ an alternating or skew-symmetric linear function

$$\omega_x : (T_xM)^q := T_xM \times \dots T_xM \longrightarrow \mathbb{R},$$

where by "alternating" or "skew-symmetric" we mean for each permutation σ of the set $\{1, 2, \dots, q\}$, i.e., $\sigma \in S_q$ and $\nu_1, \nu_2, \dots, \nu_q \in T_xM$, there holds

$$\omega_x(\nu_{\sigma(1)}, \nu_{\sigma(2)}, \dots, \nu_{\sigma(q)}) = (sgn\, \sigma)\omega_x(\nu_1, \nu_2, \dots, \nu_q).$$

We sometimes refer to alternating q-linear function on a linear space V a q-covector on V. So, for example, a 1-form on M is a mapping between a point x in the manifold into the dual space T_xM^*.

Next, to introduce the wedge product between covectors, we recall a (p, q)-shuffle is a permutation $\sigma \in S_{p+q}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q).$$

Definition B.7. *The* **wedge product** *of a p-covector α and q-covector β on a vector space V is the $(p+q)$-linear function*

$$(\alpha \wedge \beta)(\nu_1, \nu_2, \dots, \nu_{p+q}) = \Sigma(sgn\, \sigma)\alpha(\nu_{\sigma(1)}, \nu_{\sigma(2)}, \dots, \nu_{\sigma(p)})\beta(\nu_{\sigma(p+1)}, \nu_{\sigma(p+2)}, \dots, \nu_{\sigma(p+q)}),$$

where the sum is taken over all (p, q)-shuffles.

So for example, if α and β are 1-covectors, then

$$(\alpha \wedge \beta)(\nu_1, \nu_2) = \alpha(\nu_1)\beta(\nu_2) - \alpha(\nu_2)\beta(\nu_1).$$

In addition, the wedge product $\alpha \wedge \beta$ is a $p+q$-covector, and $\wedge$ as a binary operation, is bilinear, associative, and it is anti-commutative, i.e.,

$$\alpha \wedge \beta = (-1)^{deg(\alpha)deg(\beta)}\beta \wedge \alpha.$$

Note that a 0-covector is a constant and a 1-covector is a linear functional. And as the collection of all q-covectors forms a natural vector space, which we denote by $A_q(V)$, we see that $A_0(V) = \mathbb{R}$ and $A_1(\mathbb{R}) = V^*$, the dual space of V. In addition, we can use a basis of the vector space $A_1(V)$ to form a basis for $A_q(V)$, but we pause for notation prior to stating this result. Firstly, recall that a q-tuple of integers $I = (i_1, i_2, \dots, i_q)$ is called a **multi-index**, and if $i_1 \leq i_2 \leq \cdots \leq i_q$, then we say I is **ascending**. If the inequalities are all strict, then we say I is **strictly ascending**. In differential geometry, it is customary to write α^I as

$$\alpha^I = \alpha^{i_1} \wedge \alpha^{i_2} \wedge \cdots \wedge \alpha^{i_q}.$$

With this notation in mind, we have the following useful result.

Proposition B.2. *If $\alpha^1, \alpha^2, \ldots, \alpha^n$ forms a basis for $A_1(V)$, then a basis for the space of q-covectors on V, $A_q(V)$ is given by*

$$\Big\{\alpha^{i_1} \wedge \alpha^{i_2} \wedge \cdots \wedge \alpha^{i_q} \,:\, i_1 < i_2 \cdots < i_q < n\Big\}.$$

As noted earlier, given a point $x \in M$ in a coordinate chart $(U, \phi) = (U, x^1, x^2, \ldots, x^n)$, a basis for the tangent space T_xM is given by

$$\Big(\frac{\partial}{\partial x_1}\Big)_x, \Big(\frac{\partial}{\partial x_2}\Big)_x, \ldots, \Big(\frac{\partial}{\partial x_n}\Big)_x,$$

and dx_x^1, $dx_x^2, \ldots, dx_x^n$, which satisfies (B.1) forms a dual basis for the cotangent space $T_xM^* = A_1(T_xM)$. So, by Proposition B.2, for each $x \in M$, ω_x can be expressed as a linear combination of the form

$$\omega_x = \sum_I a_I(x)\, d_x^{i_1} \wedge d_x^{i_2} \wedge \cdots \wedge d_x^{i_q}. \tag{B.2}$$

We say the q-form is **of class** C^k or, respectively, **smooth** if M has an atlas $\{U_\alpha, \phi_\alpha\}$ such that on each U_α, the coefficients, $a_I : U_\alpha \longrightarrow \mathbb{R}$ in the expansion (B.2) of ω_x, are C^i or, respectively, smooth. And when referring to a **differential** q**-form**, we mean a smooth q-form on a manifold. For the most part, we will always deal with smooth differential q-forms but will often drop the term "smooth" throughout.

We define a **frame of differential** q**-forms** on an open set U of M to be a collection of differential q-forms $\omega_1, \omega_2, \ldots, \omega_r$ on U such that, at each point $x \in U$, the q-covectors $(\omega_1)_x, (\omega_2)_x, \ldots, (\omega_r)_x$ form a basis for the vector space $A_q(T_xM)$.

Remark B.2. *For example, on a given coordinate chart $(U, \phi) = (U, x^1, x^2, \ldots, x^n)$, the q-forms*

$$dx_x^{i_1} \wedge dx_x^{i_2} \wedge \cdots \wedge dx_x^{i_q},\ 1 \le i_1 < i_2 < \cdots < i_q \le n,$$

constitute a frame of differential q-forms on U.

From the definition of the cotangent bundle, we may view a 1**-form** on M as a mapping $\eta : M \longrightarrow T^*M$ such that for any $x \in M$, $\eta(x) \in T_xM^*$. Since M and T^*M are smooth manifolds, we can make sense of 1-forms of class C^k and C^∞.

For a function f of class C^k on M, we define the 1-form df as follows. For $x \in M$ and $X \in T_xM$,

$$df(x)X = X(f).$$

Then df is a 1-form of class C^{k-1}. For an integer $1 \le q \le n$, let $\bigwedge^q T_xM^*$ denote the vector space of skew-symmetric q-linear forms on T_xM. Given a chart (U, ϕ) of M at the point x, of associated coordinates x^i, then $\{dx_x^{i_1} \wedge dx_x^{i_2} \wedge \cdots \wedge dx_x^{i_q}\}_{i_1 < i_2 < \ldots < i_q}$ is a basis of $\bigwedge^q T_xM^*$ according to Proposition B.2, where dx_x^i's are defined as in (B.1).

We can then consider $\bigwedge^q(M)$, the disjoint union of the $\bigwedge^q T_xM^*$ over points $x \in M$, which naturally inherits a smooth structure and thus forms a smooth manifold via similar constructions as above. The dimension of this smooth manifold is $n + C_n^q$, where

$$C_n^q = \frac{n!}{q!(n-q)!}.$$

Alternatively, we can denote $\bigwedge^q T_xM^*$ and $\bigwedge^q(M)$ in terms of $\Omega^k(M)$ and $\Omega^*(M)$ as follows. That is, we let Ω^k be the vector space of differential k-forms on M, and we set

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M).$$

Remark B.3. *Recall that if R is a field, then an R-***module** *is precisely a vector space over R, and an R-module with a basis is said to be* **free** *and if the basis is finite with n elements, then we say the free R-module has* **rank** *n. For example, if (U, ϕ) is a coordinate chart on M, then $\Omega^k(U)$ is a free module over $C^\infty(U)$ of rank $\binom{n}{k}$.*

An algebra A is said to be **graded** *if it can be written as a direct sum $A = \bigoplus_{k=0}^\infty A_k$ of vector spaces A_k such that under multiplication, $A_k \cdot A_\ell \subset A_{k+\ell}$.*

Definition B.8. *A mapping $\eta : M \longrightarrow \bigwedge^q(M)$ is called an* **exterior form of degree** *q, or just an exterior q-form, if for any $x \in M$, $\eta(x) \in \bigwedge^q T_xM^*$.*

A map $\eta : M \longrightarrow \bigwedge^q(M)$ is an **exterior form of degree** q, or just an exterior q-form, if for any $x \in M$, $\eta(x) \in \bigwedge^q T_xM^*$. Again, the notion of an exterior q-form of class C^k and C^∞ make sense. To express η in local coordinates, consider some chart (U, ϕ) of M. Then a q-form η of class C^k can be expressed in (U, ϕ) by

$$\eta = \sum_{i_1 < \ldots < i_q} \eta_{i_1 \ldots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}.$$

The exterior derivative of η, denoted by $d\eta$, is the exterior $(q+1)$-form of class C^{k-1} whose expression in (U, ϕ) is given by

$$d\eta = \sum_{i_1 < \ldots < i_q} d\eta_{i_1 \ldots i_q} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q}$$

B.4 Pullback of Differential Forms

B.5 Riemannian manifolds

Definition B.9. *A Riemannian metric on a smooth manifold M is a correspondence which associates to each point $x \in M$ an inner product $\langle\, , \rangle_x$, i.e., symmetric, bilinear, positive-definite form, on the tangent space T_xM; moreover, the assignment $x \mapsto \langle\, , \rangle_x$ is smooth in the following sense: if X and Y are smooth vector fields on M, then $\langle X_x, Y_x \rangle_x$ is a C^∞ function on M.*

We will often drop the point x is our inner product notation. Interestingly, thanks to our definition of a manifold and a partition of unity argument, every manifold admits a Riemannian metric, which we call a Riemannian manifold.

Definition B.10. *A Riemannian manifold is a pair* $(M, \langle\, , \,\rangle)$ *consisting of a manifold* M *together with a Riemannian metric* $\langle\, , \,\rangle$ *on* M.

To be continued....

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